STA 312F2007 Solutions to Quiz 11

1. (a) Partition $\mathbf{\Sigma}$ into

$$\begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ & \mathbf{\Sigma}_{22} \end{bmatrix}$$

Entries in $\mathbf{\Sigma}_{11}$:

$$\begin{aligned}
Var(X_1) &= Var(F_1 + e_1) = Var(F_1) + Var(e_1) = \phi_{11} + \psi_1 \\
Var(X_2) &= Var(\lambda_2 F_1 + e_2) = \lambda_2^2 Var(F_1) + Var(e_2) = \lambda_2^2 \phi_{11} + \psi_2 \\
Var(X_3) &= Var(\lambda_3 F_1 + e_3) = \lambda_3^2 Var(F_1) + Var(e_3) = \lambda_3^2 \phi_{11} + \psi_3 \\
Cov(X_1, X_2) &= Cov(F_1 + e_1, \lambda_2 F_1 + e_2) = \lambda_2 Var(F_1) = \lambda_2 \phi_{11} \\
Cov(X_1, X_3) &= Cov(F_1 + e_1, \lambda_3 F_1 + e_3) = \lambda_3 Var(F_1) = \lambda_3 \phi_{11} \\
Cov(X_2, X_3) &= Cov(\lambda_2 F_1 + e_2, \lambda_3 F_1 + e_3) = \lambda_2 \lambda_3 Var(F_1) = \lambda_2 \lambda_3 \phi_{11}
\end{aligned}$$

Entries in $\mathbf{\Sigma}_{12}$:

$$\begin{aligned}
Cov(X_1, X_4) &= Cov(F_1 + e_1, F_2 + e_4) = Cov(F_1, F_2) = \phi_{12} \\
Cov(X_1, X_5) &= Cov(F_1 + e_1, \lambda_5 F_2 + e_5) = \lambda_5 Cov(F_1, F_2) = \lambda_5 \phi_{12} \\
Cov(X_1, X_6) &= Cov(F_1 + e_1, \lambda_6 F_2 + e_6) = \lambda_6 Cov(F_1, F_2) = \lambda_6 \phi_{12} \\
Cov(X_2, X_4) &= Cov(\lambda_2 F_1 + e_2, F_2 + e_4) = \lambda_2 Cov(F_1, F_2) = \lambda_2 \phi_{12} \\
Cov(X_2, X_5) &= Cov(\lambda_2 F_1 + e_2, \lambda_5 F_2 + e_5) = \lambda_2 \lambda_5 Cov(F_1, F_2) = \lambda_2 \lambda_5 \phi_{12} \\
Cov(X_2, X_6) &= Cov(\lambda_2 F_1 + e_2, \lambda_6 F_2 + e_6) = \lambda_2 \lambda_6 Cov(F_1, F_2) = \lambda_2 \lambda_6 \phi_{12} \\
Cov(X_3, X_4) &= Cov(\lambda_3 F_1 + e_3, F_2 + e_4) = \lambda_3 Cov(F_1, F_2) = \lambda_3 \phi_{12} \\
Cov(X_3, X_5) &= Cov(\lambda_3 F_1 + e_3, \lambda_5 F_2 + e_5) = \lambda_3 \lambda_5 Cov(F_1, F_2) = \lambda_3 \lambda_5 \phi_{12} \\
Cov(X_3, X_6) &= Cov(\lambda_3 F_1 + e_3, \lambda_6 F_2 + e_6) = \lambda_3 \lambda_6 Cov(F_1, F_2) = \lambda_3 \lambda_6 \phi_{12}
\end{aligned}$$

Note that entries in $\mathbf{\Sigma}_{22}$ are similar to those in $\mathbf{\Sigma}_{11}$. So, we have

$$\mathbf{\Sigma} = \left[\begin{array}{ccc|ccc}
\phi_{11} + \psi_1 & \lambda_2 \phi_{11} & \lambda_3 \phi_{11} & \phi_{12} & \lambda_5 \phi_{12} & \lambda_6 \phi_{12} \\
 & \lambda_2^2 \phi_{11} + \psi_2 & \lambda_2 \lambda_3 \phi_{11} & \lambda_2 \phi_{12} & \lambda_2 \lambda_5 \phi_{12} & \lambda_2 \lambda_6 \phi_{12} \\
 & & \lambda_3^2 \phi_{11} + \psi_3 & \lambda_3 \phi_{12} & \lambda_3 \lambda_5 \phi_{12} & \lambda_3 \lambda_6 \phi_{12} \\
\hline
 & & & \phi_{22} + \psi_4 & \lambda_5 \phi_{22} & \lambda_6 \phi_{22} \\
 & & & & \lambda_5^2 \phi_{22} + \psi_5 & \lambda_5 \lambda_6 \phi_{22} \\
 & & & & & \lambda_6^2 \phi_{22} + \psi_6
\end{array}\right]$$

(b) Yes, the model is identified.

First let us consider the identifying equations associated to $\boldsymbol{\Sigma}_{11}$:

$$\begin{aligned}
\sigma_{11} &= \phi_{11} + \psi_1 & - \quad (1)\\
\sigma_{12} &= \lambda_2\phi_{11} & - \quad (2)\\
\sigma_{13} &= \lambda_3\phi_{11} & - \quad (3)\\
\sigma_{22} &= \lambda_2^2\phi_{11} + \psi_2 & - \quad (4)\\
\sigma_{23} &= \lambda_2\lambda_3\phi_{11} & - \quad (5)\\
\sigma_{33} &= \lambda_3^2\phi_{11} + \psi_3 & - \quad (6)
\end{aligned}$$

$$\begin{aligned}
\frac{(2)(3)}{(5)} &: \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}} = \phi_{11}\\
(2) &: \lambda_2 = \frac{\sigma_{12}}{\phi_{11}}\\
(3) &: \lambda_3 = \frac{\sigma_{13}}{\phi_{11}}\\
(1) &: \psi_1 = \sigma_{11} - \phi_{11}\\
(4) &: \psi_2 = \sigma_{22} - \lambda_2^2\phi_{11}\\
(6) &: \psi_3 = \sigma_{33} - \lambda_3^2\phi_{11}
\end{aligned}$$

A unique solution is found for the parameters involved. Note that the parameters in $\boldsymbol{\Sigma}_{22}$ are identified as well in the same way.

Now the only parameter left to be identified is ϕ_{12}. In $\boldsymbol{\Sigma}_{12}$, $\phi_{12} = \sigma_{14}$, so it is identified.

Overall, a unique solution is found, thus the model is identified.

2. Yes, it would be identified.