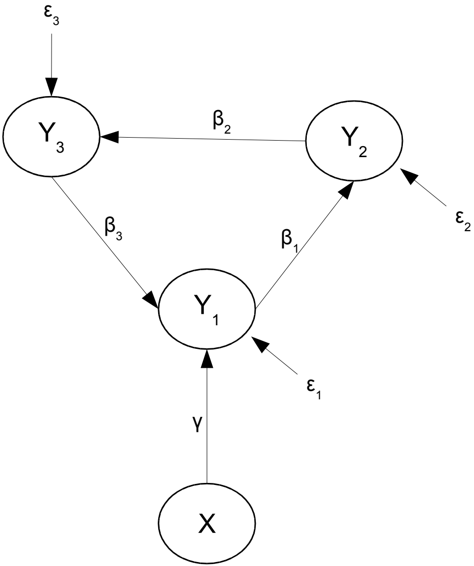
The Pinwheel Model[1]

STA431 Winter/Spring 2013

[1] See last slide for copyright information.

A Cyclic Model

Not an *Acyclic* model



Model equations

Assume all variances positive etc.

$$
\begin{aligned}
Y_1 &= \beta_3 Y_3 + \gamma X + \epsilon_1 \\
Y_2 &= \beta_1 Y_1 + \epsilon_2 \\
Y_3 &= \beta_2 Y_2 + \epsilon_3
\end{aligned}
$$

In matrix terms:

$$
\left(\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \end{array} \right) = \left(\begin{array}{rrr} 0 & 0 & \beta_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{array} \right) \left(\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \end{array} \right) + \left(\begin{array}{c} \gamma \\ 0 \\ 0 \end{array} \right) X + \left(\begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{array} \right)
$$

To get $V(\mathbf{Y})$

$$
\begin{aligned}
& \mathbf{Y} = \boldsymbol{\beta}\mathbf{Y} + \boldsymbol{\Gamma}\mathbf{X} + \boldsymbol{\epsilon} \\
\Rightarrow \quad & \mathbf{Y} - \boldsymbol{\beta}\mathbf{Y} = \boldsymbol{\Gamma}\mathbf{X} + \boldsymbol{\epsilon} \\
\Rightarrow \quad & \mathbf{I}\mathbf{Y} - \boldsymbol{\beta}\mathbf{Y} = \boldsymbol{\Gamma}\mathbf{X} + \boldsymbol{\epsilon} \\
\Rightarrow \quad & (\mathbf{I} - \boldsymbol{\beta})\mathbf{Y} = \boldsymbol{\Gamma}\mathbf{X} + \boldsymbol{\epsilon}
\end{aligned}
$$

$(\mathbf{I} - \boldsymbol{\beta})^{-1}$ exists when $|\mathbf{I} - \boldsymbol{\beta}| \neq 0$

$$
\mathbf{I} - \boldsymbol{\beta} = \begin{pmatrix} 1 & 0 & -\beta_3 \\ -\beta_1 & 1 & 0 \\ 0 & -\beta_2 & 1 \end{pmatrix}
$$

Calculate the determinant using Sage

```
sem = 'http://www.utstat.toronto.edu/~brunner/openSEM/sage/sem.sage'
load(sem)
B = ZeroMatrix(3,3)
B[0,2] = var('beta3'); B[1,0] = var('beta1'); B[2,1] = var('beta2')
ImB = IdentityMatrix(3)-B
show( ImB.determinant() )
```

$$-\beta_1\beta_2\beta_3 + 1$$

So the inverse will exist unless $\beta_1\beta_2\beta_3 = 1$.

Solve for Y_3

Starting with the model equations

$$\begin{aligned} Y_1 &= \beta_3 Y_3 + \gamma X + \epsilon_1 \\ Y_2 &= \beta_1 Y_1 + \epsilon_2 \\ Y_3 &= \beta_2 Y_2 + \epsilon_3 \end{aligned}$$

$$\begin{aligned} & Y_3 = \beta_1\beta_2\beta_3 Y_3 + \beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3 \\ \Rightarrow \quad & Y_3(1 - \beta_1\beta_2\beta_3) = \beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3 \end{aligned}$$

What happens if $(\mathbf{I} - \boldsymbol{\beta})^{-1}$ does not exist (and $\gamma \neq 0$)?

If $\beta_1\beta_2\beta_3 = 1$

Meaning that $(\mathbf{I} - \boldsymbol{\beta})^{-1}$ does not exist

$$
\begin{array}{rl}
& Y_3(1 - \beta_1\beta_2\beta_3) = \beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3 \\
\Rightarrow & 0 = \beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3 \\
\Rightarrow & E(X \cdot 0) = E\left(X(\beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3)\right) \\
\Rightarrow & 0 = \beta_1\beta_2\gamma E(X^2) + 0 \\
\Rightarrow & \beta_1\beta_2\gamma\phi = 0
\end{array}
$$

with β_1, β_2, γ and ϕ all non-zero.

So $\beta_1\beta_2\beta_3 = 1$ contradicts the model.

Under the assumptions of the pinwheel model

- $(\mathbf{I} - \boldsymbol{\beta})^{-1}$ exists.
- $\beta_1\beta_2\beta_3 \neq 1$.
- The surface $\beta_1\beta_2\beta_3 = 1$ forms a *hole* in the parameter space.

Covariance matrix of the factors: $\boldsymbol{\Phi}$

Factors are X, Y_1, Y_2, Y_3

$$\left(\begin{array}{cccc}
\phi & -\frac{\gamma\phi}{\beta_1\beta_2\beta_3-1} & -\frac{\beta_1\gamma\phi}{\beta_1\beta_2\beta_3-1} & -\frac{\beta_1\beta_2\gamma\phi}{\beta_1\beta_2\beta_3-1} \\
 & \frac{\beta_2^2\beta_3^2\psi_2+\beta_3^2\psi_3+\gamma^2\phi+\psi_1}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_1\beta_3^2\psi_3+\beta_1\gamma^2\phi+\beta_2\beta_3\psi_2+\beta_1\psi_1}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_1\beta_2\gamma^2\phi+\beta_2^2\beta_3\psi_2+\beta_1\beta_2\psi_1+\beta_3\psi_3}{(\beta_1\beta_2\beta_3-1)^2} \\
 & & \frac{\beta_1^2\beta_3^2\psi_3+\beta_1^2\gamma^2\phi+\beta_1^2\psi_1+\psi_2}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_1^2\beta_2\gamma^2\phi+\beta_1^2\beta_2\psi_1+\beta_1\beta_3\psi_3+\beta_2\psi_2}{(\beta_1\beta_2\beta_3-1)^2} \\
 & & & \frac{\beta_1^2\beta_2^2\gamma^2\phi+\beta_1^2\beta_2^2\psi_1+\beta_2^2\psi_2+\psi_3}{(\beta_1\beta_2\beta_3-1)^2}
\end{array}\right)$$

- $\phi = \phi_{11}$, $\beta_1 = \frac{\phi_{13}}{\phi_{12}}$ and $\beta_2 = \frac{\phi_{14}}{\phi_{13}}$ are easy.
- But then?

Solutions exist provided $\beta_1, \beta_2, \beta_3$ are all non-zero.

Using Sage ...

$$
\begin{aligned}
\beta_3 &= \frac{\phi_{12}\phi_{13}\phi_{23} - \phi_{13}^2\phi_{22}}{\phi_{12}\phi_{14}\phi_{33} - \phi_{13}\phi_{14}\phi_{23}} \\
\gamma &= \frac{\phi_{12}^2\phi_{33} - 2\,\phi_{12}\phi_{13}\phi_{23} + \phi_{13}^2\phi_{22}}{\phi_{11}\phi_{12}\phi_{33} - \phi_{11}\phi_{13}\phi_{23}} \\
\psi_3 &= \frac{(\phi_{13}\phi_{44} - \phi_{14}\phi_{34})(\phi_{12}^2\phi_{33} - 2\,\phi_{12}\phi_{13}\phi_{23} + \phi_{13}^2\phi_{22})}{(\phi_{12}\phi_{33} - \phi_{13}\phi_{23})\phi_{12}\phi_{13}} \\
\psi_2 &= \frac{\phi_{12}^2\phi_{33} - 2\,\phi_{12}\phi_{13}\phi_{23} + \phi_{13}^2\phi_{22}}{\phi_{12}^2} \\
\psi_1 &= \beta_1^2\beta_2^2\beta_3^2\phi_{22} - \beta_2^2\beta_3^2\psi_2 - 2\,\beta_1\beta_2\beta_3\phi_{22} - \beta_3^2\psi_3 - \gamma^2\phi + \phi_{22}
\end{aligned}
$$

Parameters of the pinwheel model are identifiable

- Even though it does not fit any known rules
- And the proof is very difficult.

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