STA 413 Formulas

Distribution	$f(x)$	$M(t)$	$E(X)$	$Var(X)$
Bernoulli	$\theta^x(1-\theta)^{1-x}I(x=0,1)$	$\theta e^t+1-\theta$	θ	$\theta(1-\theta)$
Binomial	$\binom{m}{x}\theta^x(1-\theta)^{m-x}I(x=0,\ldots,m)$	$(\theta e^t+1-\theta)^m$	$m\theta$	$m\theta(1-\theta)$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}I(x=0,1,\ldots)$	$e^{\lambda(e^t-1)}$	λ	λ
Geometric	$\theta(1-\theta)^{x-1}I(x=1,2,\ldots)$	$\theta(e^{-t}+\theta-1)^{-1}$	$\frac{1}{\theta}$	$\frac{1-\theta}{\theta^2}$
Exponential	$\frac{1}{\theta}e^{-x/\theta}I(x>0)$	$(1-\theta t)^{-1}$	θ	θ^2
Gamma	$\frac{1}{\beta^\alpha\Gamma(\alpha)}e^{-x/\beta}x^{\alpha-1}I(x>0)$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
Chi-square	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)}e^{-x/2}x^{\nu/2-1}I(x>0)$	$(1-2t)^{-\nu/2}$	ν	2ν
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$e^{\mu t+\frac{1}{2}\sigma^2t^2}$	μ	σ^2
Uniform	$\frac{1}{\beta-\alpha}I(\alpha\le x\le\beta)$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$	$\frac{\alpha+\beta}{2}$	$\frac{(\beta-\alpha)^2}{12}$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\sum_{k=r}^{\infty}a^k=\frac{a^r}{1-a}\text{ for }0<a<1$$

$$E(g(X))\ge aPr\{g(X)\ge a\}\text{ for }g(x)\ge 0$$

$$Pr\{|X-\mu|\ge k\sigma\}\le\frac{1}{k^2}$$

$$F_{Y_1}(y)=1-[1-F(y)]^n$$

$$f_{Y_1}(y)=n[1-F(y)]^{n-1}f(y)$$

$$F_{Y_n}(y)=[F(y)]^n$$

$$f_{Y_n}(y)=n[F(y)]^{n-1}f(y)$$

$$S^2=\frac{\sum_{i=1}^n(X_i-\overline{X})^2}{n-1}=\frac{1}{n-1}\left(\sum_{i=1}^n X_i^2-n\overline{X}^2\right)$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$$

$$I(\theta)=-E\left(\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right)=E\left(\frac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2$$

$$Var(T)\le\frac{[k'(\theta)]^2}{n\,I(\theta)}$$

$$\overline{x}_n\pm\frac{\widehat{\sigma}_n}{\sqrt{n}}z_{\alpha/2}$$

$$\widehat{\theta}_n\pm\frac{z_{\alpha/2}}{\sqrt{n\,I(\widehat{\theta}_n)}}$$

1. Convergence Definitions (All quantities in boldface are vectors in $\mathbb{R}^m$ unless otherwise stated)

   - $\star$ $\mathbf{X}_n \stackrel{a.s.}{\to} \mathbf{X}$ means $P\{c: \lim_{n\to\infty} \mathbf{X}_n(c) = \mathbf{X}(c)\} = 1$.
   - $\star$ $\mathbf{X}_n \stackrel{P}{\to} \mathbf{X}$ means for all $\epsilon > 0$, $\lim_{n\to\infty} P\{|\mathbf{X}_n - \mathbf{X}| < \epsilon\} = 1$.
   - $\star$ $\mathbf{X}_n \stackrel{d}{\to} \mathbf{X}$ means for every continuity point $\mathbf{x}$ of $F_{\mathbf{X}}$, $\lim_{n\to\infty} F_{\mathbf{X}_n}(\mathbf{x}) = F_{\mathbf{X}}(\mathbf{x})$.

2. $\mathbf{X}_n \stackrel{a.s.}{\to} \mathbf{X} \Rightarrow \mathbf{X}_n \stackrel{P}{\to} \mathbf{X} \Rightarrow \mathbf{X}_n \stackrel{d}{\to} \mathbf{X}$.

3. $\mathbf{X}_n \stackrel{d}{\to} \mathbf{c} \Rightarrow \mathbf{X}_n \stackrel{P}{\to} \mathbf{c}$, where $\mathbf{c}$ is a vector of constants.

4. Variance Rule: Let $T_1, T_2, \ldots$ be a sequence of real-valued random variables. If $\lim_{n\to\infty} E(T_n) = \theta$ and $\lim_{n\to\infty} Var(T_n) = 0$, then $T_n \stackrel{P}{\to} \theta$.

5. Law of Large Numbers: Let $\mathbf{X}_1, \ldots \mathbf{X}_n$ be i.i.d. random vectors with finite first moment $E(\mathbf{X}_1) = \boldsymbol{\mu}$. The Strong Law of Large Numbers (SLLN) says $\overline{\mathbf{X}}_n \stackrel{a.s.}{\to} \boldsymbol{\mu}$. The Weak Law of Large Numbers (WLLN), which follows from the Strong Law, says $\overline{\mathbf{X}}_n \stackrel{P}{\to} \boldsymbol{\mu}$.

6. Central Limit Theorem: Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. random vectors with expected value vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Then $\sqrt{n}(\overline{\mathbf{X}}_n - \boldsymbol{\mu})$ converges in distribution to a multivariate normal with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$.

7. Slutsky Theorems for Convergence in Distribution:

   (a) If $\mathbf{X}_n \in \mathbb{R}^m$, $\mathbf{X}_n \stackrel{d}{\to} \mathbf{X}$ and if $g: \mathbb{R}^m \to \mathbb{R}^q$ (where $q \leq m$) is continuous except possibly on a set C with $P(\mathbf{X} \in C) = 0$, then $g(\mathbf{X}_n) \stackrel{d}{\to} g(\mathbf{X})$.

   (b) If $\mathbf{X}_n \stackrel{d}{\to} \mathbf{X}$ and $(\mathbf{X}_n - \mathbf{Y}_n) \stackrel{P}{\to} 0$, then $\mathbf{Y}_n \stackrel{d}{\to} \mathbf{X}$.

   (c) If $\mathbf{X}_n \in \mathbb{R}^d$, $\mathbf{Y}_n \in \mathbb{R}^k$, $\mathbf{X}_n \stackrel{d}{\to} \mathbf{X}$ and $\mathbf{Y}_n \stackrel{P}{\to} \mathbf{c}$, then

$$\begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \end{pmatrix} \stackrel{d}{\to} \begin{pmatrix} \mathbf{X} \\ \mathbf{c} \end{pmatrix}$$

8. Slutsky Theorems for Convergence in Probability:

   (a) If $\mathbf{X}_n \in \mathbb{R}^m$, $\mathbf{X}_n \stackrel{P}{\to} \mathbf{X}$ and if $g: \mathbb{R}^m \to \mathbb{R}^q$ (where $q \leq m$) is continuous except possibly on a set C with $P(\mathbf{X} \in C) = 0$, then $g(\mathbf{X}_n) \stackrel{P}{\to} g(\mathbf{X})$.

   (b) If $\mathbf{X}_n \stackrel{P}{\to} \mathbf{X}$ and $(\mathbf{X}_n - \mathbf{Y}_n) \stackrel{P}{\to} 0$, then $\mathbf{Y}_n \stackrel{P}{\to} \mathbf{X}$.

   (c) If $\mathbf{X}_n \in \mathbb{R}^d$, $\mathbf{Y}_n \in \mathbb{R}^k$, $\mathbf{X}_n \stackrel{P}{\to} \mathbf{X}$ and $\mathbf{Y}_n \stackrel{P}{\to} \mathbf{Y}$, then

$$\begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \end{pmatrix} \stackrel{P}{\to} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$$

9. Taylor's Theorem (Just for two terms plus remainder): Let $g(x)$ be a function with $g''(x)$ continuous at $x = \theta$. Then

$$g(x) = g(\theta) + g'(\theta)(x - \theta) + \frac{g''(\theta^*)(x - \theta)^2}{2!},$$

where θ^* is between x and θ.

10. Univariate Delta Method: Let $\sqrt{n}(X_n - \theta) \xrightarrow{d} X$, and let $g(x)$ be a function with $g'(\theta) \neq 0$ and $g''(x)$ continuous at $x = \theta$. Then

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} g'(\theta)X.$$

In particular, $\sqrt{n}(g(\overline{X}_n) - g(\mu)) \xrightarrow{d} Y \sim N(0, g'(\mu)^2\sigma^2)$.

11. Multivariate Delta Method: Let $g : \mathbb{R}^d \to \mathbb{R}^k$ be such that the elements of $\dot{\text{g}}(\mathbf{x}) = \left[\frac{\partial g_i}{\partial x_j}\right]_{k \times d}$ are continuous in a neighborhood of $\boldsymbol{\theta} \in \mathbb{R}^d$. If $\mathbf{X}_n$ is a sequence of d-dimensional random vectors such that $\sqrt{n}(\mathbf{X}_n - \boldsymbol{\theta}) \xrightarrow{d} \mathbf{X}$, then

$$\sqrt{n}(g(\mathbf{X}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} \dot{\text{g}}(\boldsymbol{\theta})\mathbf{X}.$$

In particular, if $\sqrt{n}(\mathbf{X}_n - \boldsymbol{\theta}) \xrightarrow{d} \mathbf{X} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, then

$$\sqrt{n}(g(\mathbf{X}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} \mathbf{Y} \sim N(\mathbf{0}, \dot{\text{g}}(\boldsymbol{\theta})\boldsymbol{\Sigma}\dot{\text{g}}(\boldsymbol{\theta})').$$

12. Asymptotic Normality of the MLE: Under regularity conditions,
$\sqrt{nI(\widehat{\theta}_n)}\,(\widehat{\theta}_n - \theta) \xrightarrow{d} Z \sim N(0, 1)$

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \leq x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \geq 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998