STA 302/1001 Summer 2001 Assignment 1

Quiz on May 23d. Do this assignment in preparation for the quiz. It is not to be handed in. The rule is that if you know how to do all the problems (including *why* your answers are correct), you will do fine on the quiz.

1. In the following, assume that all the random variables involved are continuous, and that the appropriate expected values exist.
   (a) Give the definition of $E[X] = \mu_X$.
   (b) Give a convenient expression (sometimes taken for a definition) of $E[g(X)]$.
   (c) Give the definition of $Var(X) = \sigma^2_X$.
   (d) Prove $E[aX + bY] = aE[X] + bE[Y$. Do *not* assume independence. This is the only question where you have to integrate. For the others, you can just manipulate the expected value operator.
   (e) Prove $Var(Z) = E[Z^2] - (E[Z])^2$.
   (f) Give the definition of $Cov(X, Y) = \sigma_{XY}$.
   (g) Prove $Cov(X, Y) = E[XY] - E[X]E[Y]$.
   (h) Find an expression for $Var(aX + bY)$ in terms of $a, b, Var(X), Var(Y)$ and $Cov(X, Y)$.

2. Let X be Normal(μ, σ^2). Show $E[X] = \mu$ from the definition of expected value.

3. Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. You may use the fact that $\frac{\overline{X}-\mu}{s/\sqrt{n}}$ is distributed as $t(n-1)$.
   (a) Derive a $(1-\alpha) \times 100\%$ confidence interval for μ.
   (b) Show that the null hypothesis $H_0 : \mu = \mu_0$ will be rejected with the usual two-sided t-test using significance level α if and only if μ_0 is outside the $(1-\alpha) \times 100\%$ confidence interval for μ.

4. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables (that is, a random sample), whose distribution is *discrete*, and whose probability mass function $p(x; \theta)$ is known except for the parameter θ. The *maximum likelihood estimate* (MLE) of θ is the value of θ that makes the observed data $x_1, \ldots, x_n$ most likely. It is calculated by treating the joint probability mass function of $X_1, \ldots, X_n$ evaluated at $x_1, \ldots, x_n$ as a function of θ, and maximizing over θ.
   (a) Let $X_1, \ldots, X_n$ be Poisson(λ). Find the MLE of λ.
   (b) Let $X_1, \ldots, X_n$ be Binomial(n, p). Find the MLE of p.

5. For continuous distributions, we follow the same procedure, using densities instead of probability mass functions. The function we are maximizing (which is called the *likelihood function* whether the assumed distributions are discrete or continuous) is no longer the probability of obtaining the observed data, but it is similar.
   (a) Let $X_1, \ldots, X_n$ have density $\theta e^{-x\theta}$ for $x > 0$, where $\theta > 0$. Find the MLE of θ.

(b) Let $X_1, \ldots, X_n$ be normal with *fixed* (known) variance σ^2. Find the MLE of μ.

(c) Let $X_1, \ldots, X_n$ be normal with *fixed* (known) mean μ. Find the MLE of σ^2.

(d) Let $X_1, \ldots, X_n$ be normal with μ and σ^2 both unknown. Find the MLE (which is vector-valued in this case).

6. In the following, A and B are $n \times p$ matrices (of constants), C is $p \times k$, D is $p \times n$ and a, b, c are scalars. For each statement below, either prove it is true, or prove that it is not true in general by giving a counter-example.

   (a) $A + B = B + A$

   (b) $a(B + C) = aB + aC$

   (c) $AC = CA$

   (d) $(A + B)' = A' + B'$

   (e) $(AC)' = A'C'$

   (f) $(A + B)C = AC + BC$

   (g) $(AD)^{-1} = A^{-1}D^{-1}$. Assume both matrices are square.

7. That was all review. **Roughly half the points on Quiz One will come from the review part, and half will be based on the following.** Read Chapter 1; do Problems 1.5, 1.7, 1.8 1.10, 1.12 and Exercises 1.29, 1.30, 1.32, 1.33, 1.34, 1.35, 1.36, 1.41.