# The Centered Model



Sec 7.5, P. 154 does it differently

Re-parameterization: Normal examples

**Theorem** (Invariance of least squares estimation)

For the linear model $Y = X\beta + \epsilon$, let

$\alpha = g(\beta)$, where $g: \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is a 1-1 function.

Then $\hat{\alpha} = g(\hat{\beta})$

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**Proof**

$$Q(\beta) = (Y - X\beta)'(Y - X\beta)$$

Columns of $X$ are linearly independent, so

$Q(\hat{\beta}) < Q(b) \quad \forall\, b \neq \hat{\beta}$ : Minimum is unique

Because $g$ is 1-1, it has an inverse $g^{-1}$ with

$$\alpha = g(\beta) \iff \beta = g^{-1}(\alpha)$$

$$Q(\beta) = Q(g^{-1}(g(\beta))) = Q(g^{-1}(\alpha)) = Q_2(\alpha)$$

Minimize $Q_2(\alpha)$ over all $\alpha \in \mathbb{R}^{k+1}$, obtaining $\hat{\alpha}$

$$Q_2(\hat{\alpha}) \leq Q_2(a) \quad \forall\, a \in \mathbb{R}^{k+1}$$

Because (repeating) $Q_2(\hat{\alpha}) \le Q_2(a)$ $\forall\, a \in \mathbb{R}^{k+1}$,

$$Q_2(\hat{\alpha}) \le Q_2(g(\hat{\vec\beta})) = Q(g^{-1}(g(\hat{\vec\beta}))) = Q(\hat{\vec\beta})$$

$$\begin{array}{c} Q_2(\hat{\alpha}) \\ \parallel \\ Q(\underbrace{g^{-1}(\hat{\alpha})}_{b}) \end{array}$$

$Q(g^{-1}(\hat{\alpha})) < Q(\hat{\vec\beta})$ is impossible, so

$Q(g^{-1}(\hat{\alpha})) = Q(\hat{\vec\beta})$ But the minimum of $Q$ is unique, so

$$g^{-1}(\hat{\alpha}) = \hat{\vec\beta} \iff \hat{\alpha} = g(\hat{\vec\beta})$$

done

Comment: This same argument proves the invariance of MLEs

Uncentered model with intercept

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \varepsilon_i$$

Centered model with intercept

$$Y_i = \alpha_0 + \alpha_1 (x_{i1} - \bar{x}_1) + \cdots + \alpha_k (x_{ik} - \bar{x}_k) + \varepsilon_i$$

$$= \overbrace{\alpha_0 - \alpha_1 \bar{x}_1 - \cdots - \alpha_k \bar{x}_k}^{\beta_0}$$

$$+ \underset{\substack{\uparrow \\ \beta_1}}{\alpha_1} x_{i1} + \cdots + \underset{\substack{\uparrow \\ \beta_k}}{\alpha_k} x_{ik} + \varepsilon_i$$

The centered model is just a re-parameterization of the usual uncentered model, with

$$\alpha_j = \beta_j \quad \text{for } j = 1, \ldots, k, \text{ and}$$

$$\alpha_0 = \beta_0 + \beta_1 \bar{x}_1 + \cdots + \beta_k \bar{x}_k$$

$$= E(Y_i \mid x_{i1} = \bar{x}_1, \ldots, x_{ik} = \bar{x}_k)$$

leave up

By the invariance theorem,

$$\hat{\alpha}_j = \hat{\beta}_j \quad \text{for } j=1,\dots,k \text{ and}$$

$$\hat{\alpha}_0 = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}_1 + \dots + \hat{\beta}_k \bar{x}_k = \bar{Y}$$

So when you fit the centered model, the cloud of data points does not change, the least squares plane does not change — only the axes shift, and the intercept changes to $\alpha_0 = \bar{Y}$

<u>Residuals</u> (Should not change either)

Based on the usual <u>uncentered</u> model,

$$\begin{aligned}
\hat{\varepsilon}_i &= Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \dots - \hat{\beta}_k x_{ik} \\
&= Y_i - \bar{Y} + \bar{Y} - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \dots - \hat{\beta}_k x_{ik} \\
&= Y_i - \bar{Y} + \hat{\beta}_0 + \hat{\beta}_1 \bar{x}_1 + \dots + \hat{\beta}_k \bar{x}_k \\
&\qquad - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \dots - \hat{\beta}_k x_{ik} \\
&= Y_i - \bar{Y} - \hat{\beta}_1 (x_{i1} - \bar{x}_1) - \dots - \hat{\beta}_k (x_{ik} - \bar{x}_k) \\
&= Y_i - \hat{\alpha}_0 - \hat{\alpha}_1 (x_{i1} - \bar{x}_1) - \dots - \hat{\alpha}_k (x_{ik} - \bar{x}_k)
\end{aligned}$$

SAME    furthermore    can center

Furthermore can center $Y_i$ also and fit a model without an intercept: .5

$$Q(\alpha) = \sum_{i=1}^{n} (Y_i - \bar{Y} - \alpha_1(x_{i1} - \bar{x}_1) - \cdots - \alpha_p(x_{ip} - \bar{x}_p$$

Need not introduce special notation for this model: Just remember $Y = Y - \bar{Y}$ & $X = X - \bar{X}$; continue to use

$$\hat{\beta} = (X'X)^{-1}X'Y$$

What is $X'X$? $X'Y$?

So $\hat{\beta} =$

Maybe do this later.

Let's see how that would go.

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$$X_c = \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 & \cdots & x_{1k}-\bar{x}_k \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 & \cdots & x_{2k}-\bar{x}_k \\ \vdots & & & \vdots \\ x_{n1}-\bar{x}_1 & x_{n2}-\bar{x}_2 & \cdots & x_{nk}-\bar{x}_k \end{bmatrix}_{n\times k}$$

$$X_c'X_c = \begin{bmatrix} \Sigma(x_{i1}-\bar{x}_1)^2 & \Sigma(x_{i1}-\bar{x}_1)(x_{i2}-\bar{x}_2) & \cdots & \Sigma(x_{i1}-\bar{x}_1)(x_{ik}-\bar{x}_k) \\ \Sigma & \Sigma(x_{i2}-\bar{x}_2)^2 & & \\ & & & \\ & & & \Sigma(x_{ik}-\bar{x}_k)^2 \end{bmatrix}$$

$$X_c'Y_c = \begin{pmatrix} \Sigma(x_{i1}-\bar{x}_1)(y_i-\bar{y}) \\ \vdots \\ \Sigma(x_{ik}-\bar{x}_k)(y_i-\bar{y}) \end{pmatrix}$$

Divide both sides by $n-1$, get sample covariances. Useful later when $X$ is random.