STA 261s2005 Assignment 2

Do this assignment in preparation for the quiz on Wednesday, Jan. 19th. The questions are practice for the quiz, and are not to be handed in.

1. Let $X_1, \ldots, X_n$ be independent random variables with a continuous uniform distribution on $[0, \theta]$, and let $Y_n = \max(X_1, \ldots, X_n)$.

   (a) Let $F_{Y_n}(y)$ denote the cumulative distribution function of Y_n. Derive a formula for $F_{Y_n}(y)$; show your work. Make sure your formula is correct for all real y. This means you must either write it as a case function, or use indicator functions.

   (b) Find the limiting distribution of Y_n; that is, $Y_n \stackrel{P}{\rightarrow} Y$, and you are asked to find the cumulative distribution function of Y. Hint: Consider $\lim_{n\rightarrow\infty} F_{Y_n}(y)$ separately for $y \leq 0$, $0 < y < \theta$ and $y \geq \theta$.

2. When the support of a distribution depends upon one or more parameter values, the maximum and minimum are frequently useful estimators. To avoid doing calculations like those in Question 1a over and over again, let's just do them once in general, and then use the formulas.

   Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with cumulative distribution function $F(x)$ and density $f(x)$. Denote the *ordered* sample values (the "order statistics") by $Y_1 < Y_2 < \ldots < Y_n$.

   (a) Prove that the cumulative distribution function and density of the sample maximum Y_n are

   $$F_{Y_n}(y) = [F(y)]^n \text{ and } f_{Y_n}(y) = n[F(y)]^{n-1} f(y)$$

   (b) Prove that the cumulative distribution function and density of the sample minimum Y_1 are

   $$F_{Y_1}(y) = 1 - [1 - F(y)]^n \text{ and } f_{Y_1}(y) = n[1 - F(y)]^{n-1} f(y)$$

3. Let $X_1, \ldots, X_n$ be a random sample from a distribution for which the moment-generating function exists. Use moment-generating functions to show $\overline{X}_n \stackrel{d}{\rightarrow} Y$, where Y is a "degenerate" random variable with $Pr\{Y = \mu\} = 1$.

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution for which the moment-generating function exists. Use moment-generating functions and L'Hôpital's rule to prove the Central Limit Theorem (see lecture notes).

5. Read Sections 8.1-8.3 (Pages 264-269). Do Exercises 8.63, 8.64, 8.66, 8.67, 8.69. Also, on pages 230 and 231, do problems 6.65, 6.72 and 6.79.

6. Suppose the population mean number of rats in a variety store is a Poisson random variable with $\lambda = 5$.

   (a) For a single randomly selected store, what is the probability that the number of rats is less than 4?

(b) For a random sample of eighty stores, what is the probability that the sample mean number of rats is less than 4?

7. A manufacturer of automobile batteries claims that its best battery has a mean life length of 54 months, with a standard deviation of 6 months. The exact shape of the distribution of battery life lengths is not reported. A consumer group decides to buy 49 of these batteries and test them. Assume that the 49 batteries represent a random sample from the entire population of car batteries of this type.

   (a) What is the probability that a *single* randomly chosen battery will last less than 52 months? Hint: This question is impossible to answer. Why?

   (b) What is the probability that mean life length of the 49 batteries is less than 52 months?

8. In the population of Toronto kitchen sinks, the mean concentration of lead in the tap water after running the water for one minute is .657 micrograms per liter, with a standard deviation of .22. The exact shape of the distribution of lead concentration measurements is unknown. Answer these questions if possible. If it is not possible to answer the questions, say why.

   (a) What is the probability that the measurement from a *single* randomly chosen sink will be above .7 grams of lead per milliliter?

   (b) A random sample of nine sinks is tested. What is the probability that the sample mean is above .7 grams of lead per milliliter?

9. A standardized (multiple choice) test of academic achievement is normally distributed with mean μ=50 and standard deviation σ=10, for the population of North American university students. Answer these questions if possible. If it is not possible to answer the questions, say why.

   (a) What is the probability that the score of a *single* randomly chosen student will be at least 45 (that is 45 or more)?

   (b) A random sample of sixteen North American university students were given the test. What is the probability that their mean score will be at least 45 (that is 45 or more)?

10. In a survey of smoking habits reported in the *Journal of the Canadian Medical Association*, a random sample of 1250 current smokers was asked "On the average, how many cigarettes do you now smoke a day?" The mean response was $\overline{x} = 21.8$, with a sample standard deviation of $s = 5.2$. A representative of the tobacco industry says that this figure is in keeping with a long-standing trend that average daily consumption is no more than one pack (20 cigarettes) a day, because the difference between 20 and 21.8 is tiny and could have arisen by chance.

    Evaluate this claim by answering the following question. If the true population mean is really $\mu = 20$, what is the probability of getting a sample mean as large as $\overline{x} = 21.8$ or larger just by chance? Hint: It is okay to use s in place of σ in the Central Limit Theorem.

What do you conclude? Is the statement by the Industry representative a reasonable one, in light of your calculation?

11. A machine produces bolts, of which ten percent are defective. Find the probability that in a random sample of 400 bolts, 55 or more will be defective. Hint: Think of coding the data as $X_1, \ldots, X_{400}$, with $X_i = 1$ if bolt i is defective, and $X_i = 0$ if it is not defective. Translate the question into a question about the sample mean.

12. The military of a foreign country are planning to buy a troop transport plane. The plane has a payload of 5,000 kg, meaning it can carry a cargo of 5,000 kg safely. The average soldier in this country's army weighs 72 kg, with a standard deviation of 24 kg. Each soldier will take exactly 15 kg of equipment on the plane. The airplane will be equipped with seats weighing 10 kg each. The question is, how many seats should be put in the plane? There is room for 75 seats, but this would definitely overload the airplane.

    So, what is the maximum number of seats that should be put into the plane, subject to the restriction that the probability of overloading the plane must not exceed 1%? You may assume that the soldiers who will be on the plane are (roughly) a random sample of soldiers from the army.

    This is a challenging question. It is really too long for a quiz or the midterm, but if you can do this one, you can also do easier problems of this type. I will post my answer on the course Web page once I have done it and I am sure it is right.