Moment-generating Functions[1]
(Section 3.4)

STA 256: Fall 2019

[1] This slide show is an open-source document. See last slide for copyright information.

Overview

Moment-generating functions

$$M_X(t) = E(e^{Xt}) = \begin{cases} \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx \\ \\ \sum_x e^{xt} p_X(x) \end{cases}$$

- Moment-generating function may not exist for all t.
- It may not exist for any t.
- Existence in an interval containing $t = 0$ is what matters.
- Moment-generating functions exist for most of the common distributions.

Properties of moment-generating functions

- Moment-generating functions can be used to generate moments. A *moment* is a quantity like $E(X)$, $E(X^2)$, etc.
- Moment-generating functions correspond uniquely to probability distributions.
- It's sometimes easier to calculate the moment-generating function of $Y = g(X)$ and recognize it, than to obtain the distribution of Y directly.

Generating moments with the moment-generating function: Preparation

Theorem: A power series may be differentiated or integrated term by term, and the result is a power series with the same radius of convergence.

Generating moments with the moment-generating function

Using $e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$$
\begin{aligned}
M_X(t) &= E(e^{Xt}) \\
&= \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(xt)^k}{k!} \right) f_X(x)\, dx \\
&= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{(xt)^k}{k!} f_X(x)\, dx \\
&= \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} x^k f_X(x)\, dx \right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} E(X^k) \frac{t^k}{k!}
\end{aligned}
$$

Generating moments continued

$$
\begin{aligned}
M_{_X}(t) &= \sum_{k=0}^{\infty} E(X^k)\frac{t^k}{k!} \\
&= 1 + E(X)t + E(X^2)\frac{t^2}{2!} + E(X^3)\frac{t^3}{3!} + \cdots \\
M'_{_X}(t) &= 0 + E(X) + E(X^2)\frac{2t}{2!} + E(X^3)\frac{3t^2}{3!} + \cdots \\
&= E(X) + E(X^2)t + E(X^3)\frac{t^2}{2!} + E(X^4)\frac{t^3}{3!} + \cdots \\
M'_{_X}(0) &= E(X) \\
M''_{_X}(t) &= 0 + E(X^2) + E(X^3)t + E(X^4)\frac{t^2}{2!} + \cdots \\
M''_{_X}(0) &= E(X^2)
\end{aligned}
$$

And so on. To get $E(Y^k)$, differentiate $M_{_Y}(t)$, k times with respect to t, and set $t = 0$.

Example: Poisson Distribution

$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$ for $x = 0, 1, \ldots$

$$\begin{aligned} M(t) &= E(e^{Xt}) \\ &= \sum_{x=0}^{\infty} e^{xt} \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)} \end{aligned}$$

Differentiate to get moments for Poisson

$M(t) = e^{\lambda(e^t - 1)}$

$$
\begin{aligned}
M'(t) &= e^{\lambda(e^t-1)} \cdot \lambda e^t \\
&= \lambda e^{\lambda(e^t-1)+t}
\end{aligned}
$$

Set $t = 0$ and get $E(X) = \lambda$.

$$
\begin{aligned}
M''(t) &= \lambda e^{\lambda(e^t-1)+t} \cdot (\lambda e^t + 1) \\
&= e^{\lambda(e^t-1)+t} \cdot (\lambda^2 e^t + \lambda)
\end{aligned}
$$

Set $t = 0$ and get $E(X^2) = \lambda^2 + \lambda$.

So $Var(X) = E(X^2) = [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Useful properties of moment-generating functions

Use these to find distributions of *functions* of random variables

- $M_{aX}(t) = M_X(at)$
- $M_{a+X}(t) = e^{at} M_X(t)$
- If X and Y are independent, $M_{X+Y}(t) = M_X(t) \, M_Y(t)$

  Extending by induction,
- If $X_1, \ldots, X_n$ are independent,
  $M_{(\sum_{i=1}^n X_i)}(t) = \prod_{i=1}^n M_{X_i}(t)$.

Identifying Distributions using Moment-generating Functions

- Getting expected values with the MGF can be easier than direct calculation. But not always.
- Moment-generating functions can also be used to identify distributions.
- Calculate the moment-generating function of $Y = g(X)$, and if you recognize the MGF, you have the distribution of Y.
- Here's what's happening technically.
- $M_{_X}(t) = \int_{-\infty}^{\infty} e^{xt} f_{_X}(x)\, dx$ so $M_{_X}(t)$ is a function of $F_{_X}(x)$. That is, $M_{_X}(t) = g(F_{_X}(x))$.
- Uniqueness says the function g has an inverse, so that $F_{_X}(x) = g^{-1}(M_{_X}(t))$.

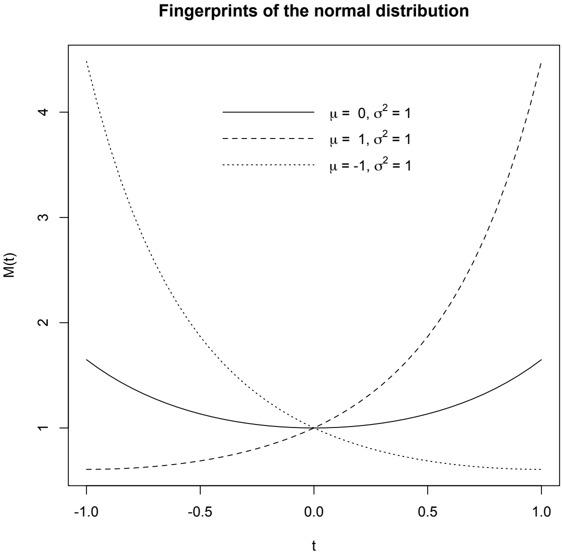
The function $M(t)$ is like a fingerprint of the probability distribution.

$Y \sim N(\mu, \sigma^2)$ if and only if $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

$Y \sim \chi^2(\nu)$ if and only if $M_Y(t) = (1 - 2t)^{-\nu/2}$ for $t < \frac{1}{2}$.

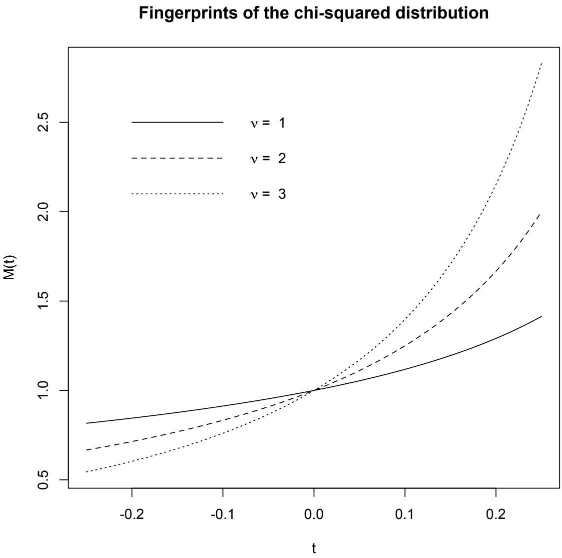
Chi-squared is a special Gamma, with $\alpha = \nu/2$ and $\lambda = \frac{1}{2}$.

Normal: $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$



Chi-squared: $M(t) = (1-2t)^{-\nu/2}$

Chi-squared is a special Gamma, with $\alpha = \nu/2$ and $\lambda = \frac{1}{2}$



Example: Sum of Poissons is Poisson

Let $X_1, \ldots, X_n$ be independent Poisson(λ_i). Let $Y = \sum_{i=1}^{n} X_i$. Find the probability distribution of Y. Recall Poisson MGF is $e^{\lambda(e^t - 1)}$.

$$
\begin{aligned}
M_Y(t) &= M_{(\sum_{i=1}^{n} X_i)}(t) \\
&= \prod_{i=1}^{n} M_{x_i}(t) \\
&= \prod_{i=1}^{n} e^{\lambda_i(e^t - 1)} \\
&= e^{(\sum_{i=1}^{n} \lambda_i)(e^t - 1)}
\end{aligned}
$$

MGF of Poisson, with $\lambda' = \sum_{i=1}^{n} \lambda_i$. Therefore, $Y \sim$ Poisson$(\sum_{i=1}^{n} \lambda_i)$.

Copyright Information