Counting Methods for Computing Probabilities[1] (Section 1.4)

STA 256: Fall 2019

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Uniform Probability on Finite Spaces

If the sample space is finite and all outcomes of an experiment are equally likely,

$$P(A) = \frac{\text{Number of ways for } A \text{ to happen}}{\text{Total number of outcomes}} = \frac{|A|}{|S|}$$

Need to count.

A Boring Example

Roll a fair die. What is the probability of an odd number?

$$\begin{aligned} P(\text{Odd}) &= P(\{1,3,5\}) \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Multiplication Principle

Also called the Fundamental Principle of Counting

If there are k experiments and the first has n_1 outcomes, the second has n_2 outcomes, etc., then there are

$$n_1 \times n_2 \times \cdots \times n_k$$

outcomes in all.

Sample Question

If there are nine horses in a race, in how many ways can they finish first, second and third?

$$9 \times 8 \times 7 = 504$$

Permutations: Ordered subsets

The notation ${}_nP_k$ is not in the text

The number of *permutations* (ordered subsets) of n objects taken k at a time is

$$
\begin{aligned}
{}_nP_k &= n \times (n-1) \times \cdots \times (n-k+1) \\
&= \frac{n!}{(n-k)!}
\end{aligned}
$$

Combinations: Unordered subsets

The vocabulary “combinations” is not in the text

The number of *combinations* (unordered subsets) of n objects taken k at a time is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example

If a club has 30 members, how many ways are there to choose a committee of 5?

$$\begin{aligned} \binom{30}{5} &= \frac{30!}{5!(30-5)!} \\ &= 142,506 \end{aligned}$$

Proof of $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Choose an unordered subset of k items from n. Then place them in order. By the Multiplication Principle,

$$\begin{aligned} {}_nP_k &= \binom{n}{k} \times k! \\ \Rightarrow \quad \frac{n!}{(n-k)!} &= \binom{n}{k} \times k! \\ \Rightarrow \quad \binom{n}{k} &= \frac{n!}{k!\,(n-k)!} \end{aligned}$$

■

Example

How many ways are there to deal a hand of 5 cards from a standard deck of 52 cards?

$$\begin{aligned} \binom{52}{5} &= \frac{52!}{5!\,47!} \\ &= 2,598,960 \end{aligned}$$

- If you could inspect one hand per second,
- It would take a little over 30 days to examine them all.
- Working 24/7.
- The point is that these counting arguments allow you to "count" objects that are so numerous you could not literally even look at them all.

Binomial Theorem

Binomial coefficients appear in the Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Multinomial Coefficients

The number of ways that n objects can be divided into ℓ subsets with k_i objects in set i, $i = 1, \ldots, \ell$ is

$$\left(\begin{array}{ccc} & n & \\ n_1 & \cdots & k_\ell \end{array}\right) = \frac{n!}{k_1! \;\cdots\; k_\ell!}$$

Multinomial Theorem

Nice to know

$$(x_1 + \cdots + x_\ell)^n = \sum_{\mathbf{k}} \binom{n}{k_1 \ \ldots \ k_\ell} x_1^{k_1} \cdots x_\ell^{k_\ell}$$

where the sum is over all non-negative integers $k_1, \ldots, k_\ell$ such that $\sum_{j=1}^{\ell} k_j = n$.

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