STA 256 ASSIGNMENT 8

① $M_{aX}(t) = E(e^{aXt}) = E(e^{X(at)}) = M_X(at)$

② $M_{X+a}(t) = E(e^{(a+X)t}) = E(e^{at}e^{Xt})$

$= e^{at}E(e^{Xt}) = e^{at}M_X(t)$

③ $M_Y(t) = E(e^{(g(X_1)+h(X_2))t}) = E(e^{g(X_1)t}e^{g(X_2)t})$

$= \sum_{x_1}\sum_{x_2} e^{g(x_1)t} e^{h(x_2)t} P_{X_1,X_2}(x_1, x_2)$

$\overset{\text{ind}}{=} \sum_{x_1}\sum_{x_2} e^{g(x_1)t} e^{h(x_2)t} P_{X_1}(x_1) P_{X_2}(x_2)$

$= \sum_{x_1} e^{g(x_1)t} P_{X_1}(x_1) \sum_{x_2} e^{h(x_2)} P_{X_2}(x_2)$

$= M_{g(X_1)}(t)\, M_{h(X_2)}(t)$

(4)

Distribution	$M_X(t)$	$E(X)$	$Var(X)$
Bernoulli (θ)	$\theta e^t + 1 - \theta$	θ	$\theta(1-\theta)$
Binomial (n, θ)	$(\theta e^t + 1 - \theta)^n$	$n\theta$	$n\theta(1-\theta)$
Poisson (λ)	$e^{\lambda(e^t - 1)}$	λ	λ
~~Uniform (α, β)~~			
Exponential (λ)	$\left(1 - \frac{t}{\lambda}\right)^{-1}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma (α, β)	$\left(1 - \frac{t}{\lambda}\right)^{-\alpha}$	α/λ	α/λ^2
Normal (μ, σ^2)	$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$	μ	σ^2
Chi-squared (ν)	$(1-2t)^{-\frac{\nu}{2}}$	ν	2ν

Binomial

$$M_X(t) = E(e^{Xt}) = \sum_{x=0}^{n} e^{xt} \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} [\theta e^t]^x (1-\theta)^{n-x}$$

$$\overset{\text{Binomial theorem}}{=} (\theta e^t + 1 - \theta)^n$$

$$E(X) = M'(0) = n(\theta e^t + 1 - \theta)^{n-1} \cdot \theta e^t \Big|_{t=0} = n\theta$$

(Continued on next page)

Q4 continued

$M''(t) = \frac{d}{dt} n\theta e^t(\theta e^t + 1 - \theta)^{n-1}$ product rule

$= n\theta\left(e^t(\theta e^t + 1-\theta)^{n-1} + e^t(n-1)(\theta e^t + 1-\theta)^{n-2} \cdot \theta e^t \right)$

$M''(0) = n\theta(1 + (n-1)\theta) = n\theta + n^2\theta^2 - n\theta^2$
$= E(X^2)$, so

$Var(X) = E(X^2) - (E(X))^2 = n\theta + \cancel{n^2\theta^2} - n\theta^2 \cancel{- n^2\theta^2}$
$= n\theta(1-\theta)$

Poisson $M_X(t) = E(e^{Xt}) = \sum_{x=0}^{\infty} e^{xt} \frac{e^{-\lambda}\lambda^x}{x!}$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$

$= e^{\lambda(e^t - 1)}$

$M'(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t$,

$M'(0) = e^0 \cdot \lambda \cdot e^0 = \lambda$

$M''(t) = \lambda \frac{d}{dt} e^{(\lambda e^t - \lambda + t)}$

$= \lambda e^{\lambda e^t - \lambda + t} \cdot (\lambda e^t + 1)$

$M''(0) = \lambda e^{\lambda - \lambda + 0}(\lambda + 1) = \lambda(\lambda+1) = E(X^2)$

$Var(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Gamma

$$M_X(t) = E(e^{Xt})$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{xt} e^{-\lambda x} x^{\alpha-1}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda - t)x} x^{\alpha-1}\, dx \quad \text{converges for } t < \lambda \text{, or}$$

$$\frac{\lambda^\alpha}{(\lambda-t)^\alpha} \underbrace{\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda-t)x} x^{\alpha-1}\, dx}_{=1}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^\alpha = \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha} = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$$

$$M'(t) = -\alpha\left(1 - \frac{t}{\lambda}\right)^{-\alpha-1} \cdot \frac{-1}{\lambda} = \frac{\alpha}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-\alpha-1}$$

$$M'(0) = E(X) = \frac{\alpha}{\lambda}$$

$$M''(t) = \frac{\alpha}{\lambda}(-\alpha-1)\left(1 - \frac{t}{\lambda}\right)^{-\alpha-2} \cdot \frac{-1}{\lambda}$$

$$= \frac{\alpha(\alpha+1)}{\lambda^2} = E(X^2) \text{, and}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{1}{\lambda^2}\left(\cancel{\alpha^2} + \alpha - \cancel{\alpha^2}\right) = \frac{\alpha}{\lambda^2}$$

Normal First Deriv MGF of $Z \sim N(0,1)$

$$M_Z(t) = E(e^{zt}) = \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \operatorname{Exp} - \tfrac{1}{2}(z^2 - 2zt + t^2 - t^2)\, dz$$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2}\, dz}_{\mu = t \quad = 1} = e^{t^2/2}$$

Now if $X \sim N(\mu, \sigma^2)$, $Z = \left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$

And $X = \sigma Z + \mu$, $M_X(t) = E(e^{(\sigma Z + \mu)t})$

$$= E(e^{Z(\sigma t)} e^{\mu t}) = e^{\mu t} E(e^{Z(\sigma t)})$$

$$= e^{\mu t} M_Z(\sigma t) = e^{\mu t} e^{\frac{1}{2}(\sigma t)^2}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$M'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot (\mu + \tfrac{1}{2}\sigma^2 2t)$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot (\mu + \sigma^2 t)$$

$$M'(0) = e^0(\mu + 0) = E(X) = \mu$$

$$M''(t) = \left[e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t)^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2} \sigma^2 \right]$$

$M''(0) = \mu^2 + \sigma^2 = E(X^2)$, so

$$Var(X) = E(X^2) - (E(X))^2 = \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

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⑤ $X \sim \text{Geometric}(\theta)$

a) $M_X(t) = E(e^{Xt}) = \sum_{x=0}^{\infty} e^{xt}(1-\theta)^x\theta$

$= \theta \sum_{x=0}^{\infty} [(1-\theta)e^t]^x$

For convergence, need $(1-\theta)e^t < 1$

$\iff e^t < (1-\theta)^{-1} \iff t < -\log(1-\theta)$

$= \theta \dfrac{1}{1-(1-\theta)e^t} = \theta(1-(1-\theta)e^t)^{-1}$ 0/T

b) $M'(t) = -\theta(1-(1-\theta)e^t)^{-2} \cdot (-1)(1-\theta)e^t$

$= \dfrac{\theta(1-\theta)e^t}{(1-(1-\theta)e^t)^2}$

$E(X) = M'(0) = \dfrac{\theta(1-\theta)}{(1-(1-\theta))^2} = \dfrac{\theta(1-\theta)}{\theta^2} = \dfrac{1-\theta}{\theta}$

(6) $X \sim N(\mu, \sigma^2)$ $\quad Z = \dfrac{X-\mu}{\sigma}$

$$M_Z(t) = E(e^{Zt}) = E\left(e^{\left(\frac{X-\mu}{\sigma}\right)t}\right)$$

$$= E\left(e^{X\left(\frac{t}{\sigma}\right)} e^{-\mu t/\sigma}\right) = e^{-\mu t/\sigma} E\left(e^{X\left(\frac{t}{\sigma}\right)}\right)$$

$$= e^{-\mu t/\sigma} M_X\left(\tfrac{t}{\sigma}\right)$$

$$= e^{-\mu t/\sigma} e^{\mu\left(\frac{t}{\sigma}\right) + \frac{1}{2}\sigma^2\left(\frac{t}{\sigma}\right)^2}$$

$$= \operatorname{Exp}\left\{-\cancel{\frac{\mu t}{\sigma}} + \cancel{\frac{\mu t}{\sigma}} + \frac{1}{2}\cancel{\sigma^2}\frac{t^2}{\cancel{\sigma^2}}\right\} = e^{\frac{1}{2}t^2}$$

MGF of $N(0,1)$

(7) $$M_Y(t) = E\left(e^{(X_1+3X_2)t}\right) = E\left(e^{X_1 t} e^{X_2 3t}\right)$$

$$\overset{\text{ind}}{=} E(e^{X_1 t}) E(e^{X_2(3t)})$$

$$= M_{X_1}(t)\, M_{X_2}(3t)$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} + e^{3\mu_2 t + \frac{1}{2}\sigma_2^2(3t)^2}$$

$$= e^{(\mu_1 + 3\mu_2)t + \frac{1}{2}(\sigma_1^2 + 9\sigma_2^2)t^2}$$

MGF of $N(\mu_1 + 3\mu_2, \sigma^2 + 9\sigma_2^2)$

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(11) $Z \sim N(0,1)$, $Y = Z^2$

$$M_Y(t) = M_{Z^2}(t) = E\left(e^{Z^2 t}\right)$$

$$= \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2t)z^2}\, dz$$

will converge if $1-2t>0$ $\Leftrightarrow$ $t<\frac{1}{2}$ only

$$= (1-2t)^{-\frac{1}{2}} \underbrace{\frac{1}{\sqrt{(1-2t)^{-1}}\,\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-2t)^{-1}}(z-0)^2}\, dz}_{=1}$$

$$= (1-2t)^{-\frac{1}{2}}$$ MGF of Chi-squared, $\nu = 1$

(same as Gamma with $\alpha = \lambda = \frac{1}{2}$)

(12) $P(X=\mu) = 1$

a) $M_X(t) = E(e^{Xt}) = \sum_x e^{xt} P_X(x)$

$= \sum_{x=\mu} e^{xt} P_X(x) = e^{\mu t} \cdot 1 = e^{\mu t}$

b) $M'(t) = \mu e^{\mu t}$

$M'(0) = \mu e^0 = \mu = E(X)$

$M''(t) = \mu^2 e^t$

$M''(0) = \mu^2 = E(X^2)$

$var(X) = E(X^2) - (E(X))^2 = \mu^2 - \mu^2 = 0 = Var(X)$

c) $M_X(t) = e^{\mu t + \frac{1}{2} 0 \cdot t^2}$

MGF of Normal$(\mu, 0)$

13. (a) Since X and Y indep., we have

$$M_Z(t) = M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

$$= \left[\frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}\right]$$

$$\times \left[\frac{1}{10}e^{-t} + \frac{4}{10}e^{2t} + \frac{1}{2}e^{3t}\right]$$

$$= \frac{1}{60}e^{0t} + \frac{2}{60}e^{t} + \frac{3}{60}e^{2t}$$

$$+ \frac{4}{60}e^{3t} + \frac{13}{60}e^{4t} + \frac{22}{60}e^{5t} + \frac{15}{60}e^{5t}.$$

Thus, from $M_Z(t)$, we have the pmf:

z	0	1	2	3	4	5	6
$p_Z(z)$	$\frac{1}{60}$	$\frac{2}{60}$	$\frac{3}{60}$	$\frac{4}{60}$	$\frac{13}{60}$	$\frac{22}{60}$	$\frac{15}{60}$

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(b) $M_Z(t) = M_{X-Y}(t) = E\left(e^{(X-Y)t}\right)$

$$= E\left(e^{Xt} \cdot e^{Y(-t)}\right)$$

$$\overset{\text{indep}}{=} E(e^{Xt})\,E(e^{Y(-t)})$$

$$= M_X(t)\, M_Y(-t)$$

$$= \left[\frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}\right] \cdot \left[\frac{1}{10}e^{t} + \frac{4}{10}e^{-2t} + \frac{1}{2}e^{-3t}\right]$$

$$= \frac{1}{12}e^{-2t} + \frac{14}{60}e^{-t} + \frac{23}{60}e^{0t} + \frac{12}{60}e^{t} + \frac{1}{60}e^{2t} + \frac{2}{60}e^{3t} + \frac{3}{60}e^{4t}$$

From $M_Z(t)$, we have the pmf of Z:

z	-2	-1	0	1	2	3	4
$p_Z(z)$	$\frac{1}{12}$	$\frac{14}{60}$	$\frac{23}{60}$	$\frac{12}{60}$	$\frac{1}{60}$	$\frac{2}{60}$	$\frac{3}{60}$

14. (a) $Z = X + Y$

X	Y	$Z = X + Y$
-1	2	$-1+2=1$
0	4	$-1+4=3$
1		$0+2=2$
		$0+4=4$
		$1+2=3$
		$1+4=5$

$\therefore$ Z has values : 1, 2, 3, 4, 5.

By independence,

$$p_Z(1) = P(X=-1, Y=2) = p_X(-1)\, p_Y(2) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$p_Z(2) = P(X=0, Y=2) = p_X(0)\, p_Y(2) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$p_Z(3) = P(X=-1, Y=4) + P(X=1, Y=2) = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{6}$$

$$p_Z(4) = P(X=0, Y=4) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$$

$$p_Z(5) = P(X=1, Y=4) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$$

That is,

z	1	2	3	4	5
$p_Z(z)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

.

(b) $M_Z(t) = M_{X+Y}(t) = M_X(t)\, M_Y(t)$ (by indep.)

$$M_X(t) = \frac{1}{3} e^{-t} + \frac{1}{3} e^{0t} + \frac{1}{3} e^{t} = \frac{1}{3}(e^{-t} + 1 + e^{t})$$

$$M_Y(t) = \frac{1}{2} e^{2t} + \frac{1}{2} e^{4t} = \frac{1}{2}(e^{2t} + e^{4t})$$

$$M_Z(t) = \frac{1}{6}\left[e^{t} + e^{3t} + e^{2t} + e^{4t} + e^{3t} + e^{5t} \right]$$

$$= \frac{1}{6}\left[e^{t} + e^{2t} + 2e^{3t} + e^{4t} + e^{5t} \right]$$

(c) From part (b), it is clear that

z	1	2	3	4	5
$p_Z(z)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$1/6$

Thus, the answer agrees with part (a).

15. (a) By independence,

$$M_Z(t) = M_X(t)\, M_Y(t)$$

$$= e^{\lambda(e^t-1)}\, e^{\lambda(e^t-1)}$$

$$= e^{2\lambda(e^t-1)} \rightarrow \text{m.g.f. of Poisson}(2\lambda)$$

(b) $P(X=r \mid Z=n) = P(X=r \mid X+Y=n)$

$$= \frac{P(X=r,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=r,\ Y=n-r)}{P(X+Y=n)}$$

$$\overset{\text{indep}}{=} \frac{P(X=r)\,P(Y=n-r)}{P(Z=n)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^r}{r!}\ \frac{e^{-\lambda}\lambda^{n-r}}{(n-r)!}}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}}$$

$$= \frac{n!}{r!\,(n-r)!} \left(\frac{1}{2}\right)^n$$

$$= \binom{n}{r} \left(\frac{1}{2}\right)^r \left(1 - \frac{1}{2}\right)^{n-r}$$

(c) $\therefore X \mid Z = n \sim \text{Binomial}(n, \theta = \frac{1}{2})$.

16. (a) $1 = \int_{-\infty}^{\infty} f(x)\,dx$

$$= k \left[\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx \right]$$

$$= k \left[e^{x} \Big|_{-\infty}^{0} - e^{-x} \Big|_{0}^{\infty} \right]$$

$$= k\,[1+1] \Rightarrow 2k = 1 \Rightarrow \boxed{k = \frac{1}{2}}$$

(b) $M_X(t) = E(e^{tX})$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \frac{1}{2} \left[\int_{-\infty}^{0} e^{tx} e^{x}\,dx + \int_{0}^{\infty} e^{tx} e^{-x}\,dx \right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{(t+1)x}\,dx + \int_{0}^{\infty} e^{-(1-t)x}\,dx\right]$$

$$= \frac{1}{2}\left[\left.\frac{e^{(t+1)x}}{t+1}\right|_{-\infty}^{0} - \left.\frac{e^{-(1-t)}}{1-t}\right|_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right], \quad \text{for } -1<t<1$$

$$= \frac{1}{2}\,\frac{1\cancel{-t}+1\cancel{+t}}{(1+t)(1-t)}, \quad |t|<1$$

$$= \frac{1}{1-t^2}, \quad \text{for } |t|<1.$$

(c) $M_X(t) = (1-t^2)^{-1}$

$M_X'(t) = -1(1-t^2)^{-2}(-2t) = 2t(1-t^2)^{-2}$

$\Rightarrow E(X) = M_X'(0) = \text{zero}.$

$M_X''(t) = 2(1-t^2)^{-2} + 2t(-2)(1-t^2)^{-3}(-2t)$

$\Rightarrow E(X^2) = M_X''(0) = 2$

$\therefore V(X) = E(X^2) - (E(X))^2 = 2 - 0^2 = \underline{\underline{2}}.$

17. (a) $M_X(t) = E(e^{tX})$

$$= E\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!} \right)$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$$

That is,

$E(X^k)$ = coefficient of $\frac{t^k}{k!}$ in Maclaurin/power series.

(b) (i) $M_X(t) = \frac{1}{1-t^2} = \sum_{k=0}^{\infty} (t^2)^k$

$$= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (2k)!$$

$\therefore E(X^{2k}) = (2k)! \Rightarrow E(X^{1024}) = 1024!$

$2k = 1024$

(ii) $M_X(t) = e^{t^2} = \sum_{k=0}^{\infty} \frac{(t^2)^k}{k!}$

$$= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \underbrace{\frac{(2k)!}{k!}}_{\text{coefficient of } \frac{t^{2k}}{(2k)!}}$$

$$\therefore\ E(X^{2k}) = \frac{(2k)!}{k!}$$

$$E(X^{1024}) = \frac{1024!}{\frac{1024}{2}!} = \frac{1024!}{512!}.$$

(iii) Note:

$$\frac{1}{1-5t} = \sum_{k=0}^{\infty} (5t)^k = \sum_{k=0}^{\infty} 5^k t^k$$

Differentiate both sides w.r. to t:

$$\frac{5}{(1-5t)^2} = \sum_{k=0}^{\infty} 5^k\, k\, t^{k-1}$$

$$\Rightarrow \frac{1}{(1-5t)^2} = \frac{1}{5}\sum_{k=0}^{\infty} 5^k\, k\, t^{k-1}$$

$$= \sum_{k=0}^{\infty} k\, 5^{k-1} \frac{t^{k-1}}{(k-1)!} \cdot (k-1)!$$

$$= \sum_{k=0}^{\infty} k\, 5^{k-1} (k-1)!\, \frac{t^{k-1}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \underbrace{5^{k-1}\, k!}_{\text{coefficient of } \frac{t^{k-1}}{(k-1)!}} \frac{t^{k-1}}{(k-1)!}$$

$$\Rightarrow E(X^{k-1}) = 5^{k-1}\, k!$$

$$E(X^{1024}) = 5^{1024}\, 1025!$$

18. (a) $S(0) = \log M_X(0) = \log(1) = \text{Zero}.$

(b) $$\frac{d}{dt} S(t) = \frac{M_X'(t)}{M_X(t)}.$$

$$\Rightarrow \frac{d}{dt} S(t)\Big|_{t=0} = \frac{M_X'(0)}{M_X(0)} = \frac{E(X)}{1} = E(X).$$

(c) $$\frac{d^2}{dt} S(t) = \frac{M_X(t)\, M_X''(t) - M_X'(t)\, M_X'(t)}{[M_X(t)]^2}$$

$$\therefore \frac{d^2}{dt^2} S(t)\Big|_{t=0} = \frac{M_X(0)\, M_X''(0) - M_X'(0)\, M_X'(0)}{[M_X(0)]^2}$$

$$= \frac{1 \times E(X^2) - E(X)\, E(X)}{1^2}$$

$$= E(X^2) - (E(X))^2$$

$$= Var(X).$$

19. Note:

$$Y = 0.5^X = e^{X \ln 0.5}$$

$$\Rightarrow Y^2 = e^{X(2\ln 0.5)}$$

$$\Rightarrow E(Y^2) = E\left[e^{X(2\ln(0.5))}\right]$$

$$= M_X(2\ln 0.5)$$

Since $X \sim N(\mu = 1, \sigma^2 = 4)$

$$\Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$= e^{t + 2t^2}$$

$$\therefore E(Y^2) = e^{2\ln 0.5 + 2(2\ln 0.5)^2}.$$

20. $M_X(t) = e^{3t + 8t^2}$ compare with $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$$\mu = 3, \quad \tfrac{1}{2}\sigma^2 = 8 \Rightarrow \sigma^2 = 16$$

$$\therefore X \sim N(\mu = 3, \sigma^2 = 16).$$

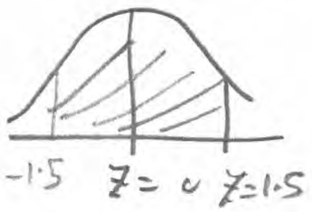
Now,

$$P(-1 < X < 9) = P\left(\frac{-1-3}{4} < Z < \frac{9-3}{4}\right)$$

$$= P(-1 < Z < 1.5)$$

$$= P(Z < 1.5) - P(Z < -1)$$

Direct from the table



$$= (1 - 0.0668) - 0.1587$$

$$= 0.7745.$$