Q1) X & Y are discrete variables.

$Cov(X,Y) = E(XY) - E(X)E(Y) = \sum_x \sum_y xy\,P(X=x, Y=y) - E(X)E(Y)$

This is where I use independence $= \sum_x x \sum_y y\, P(X=x)P(Y=y) - E(X)E(Y)$

$= \sum_x x P(X=x) \sum_y y P(Y=y) - E(X)E(Y)$

$= E(X)E(Y) - E(X)E(Y) = 0$

2) a) $Y_1 = \frac{X_1}{X_2}$, $Y_2 = X_2 \Rightarrow X_1 = Y_1 X_2 = Y_1 Y_2$

Since $y_1 = \frac{x_1}{x_2}$ and $x_1 \geq 0$, $x_2 \geq 0$, $y_1 \in [0,\infty)$ } where $f_{Y_1,Y_2} > 0$

Since $y_2 = x_2$, $y_2 \in [0,\infty)$

$$\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial (y_1 y_2)}{\partial y_1} & \frac{\partial (y_1 y_2)}{\partial y_2} \\ \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = A$$

$|\det(A)| = |y_2 - 0| = |y_2| = y_2$ since we said $y_2 \in [0,\infty)$

$f_{X_1,X_2}(x_1,x_2) \overset{X_1 \perp\!\!\!\perp X_2}{=} f_{X_1}(x_1) f_{X_2}(x_2) = e^{-x_1} e^{-x_2}$ for $x_1, x_2 \in [0,\infty)$

Applying Jacobian Formula gives:

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} e^{-y_1 y_2} e^{-y_2} y_2 = y_2 e^{-y_2(y_1+1)} & \text{for } y_1 \in [0,\infty),\ y_2 \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

b) $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_0^{\infty} y_2 e^{-y_2(y_1+1)}\,dy_2$ ...... ①

Integration by parts $\quad u = y_2 \quad dv = e^{-y_2(y_1+1)}$

$\int u\,dv = uv - \int v\,du \quad du = dy_2 \quad v = \frac{-1}{y_1+1} e^{-y_2(y_1+1)}$

$$① = \frac{-y_2}{y_1+1} e^{-y_2(y_1+1)} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{y_1+1} e^{-y_2(y_1+1)}\,dy_2$$

$$= 0 + \frac{1}{(y_1+1)^2} e^{-y_2(y_1+1)} \Big|_{\infty}^{0} = \frac{1}{(y_1+1)^2}$$

$$f_{Y_1}(y_1) = \begin{cases} \frac{1}{(y_1+1)^2} & \text{for } y_1 \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

3) a) $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = E\left(\frac{X_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_n}{n}\right)$

$= \frac{E(X_1)}{n} + \frac{E(X_2)}{n} + \cdots + \frac{E(X_n)}{n} = \frac{\mu}{n} + \frac{\mu}{n} + \cdots + \frac{\mu}{n} = \frac{n\mu}{n} = \mu$

b) $Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = Var\left(\frac{X_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_n}{n}\right)$

Due to independence $= \frac{Var(X_1)}{n^2} + \frac{Var(X_2)}{n^2} + \cdots + \frac{Var(X_n)}{n^2}$

$= \frac{n\sigma^2}{n^2} = \sigma^2/n$

c) If $X_i \sim Normal(\mu, \underset{\text{standard deviation}}{\sigma})$, $M_{X_i}(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$M_{\bar{X}}(t) = E(e^{\bar{X}t}) = E\left(e^{\frac{1}{n}\sum_{i=1}^{n} X_i t}\right) = E\left(e^{X_1\frac{t}{n} + X_2\frac{t}{n} + \cdots + X_n\frac{t}{n}}\right)$

$= E\left(e^{X_1\frac{t}{n}} e^{X_2\frac{t}{n}} e^{X_3\frac{t}{n}} \cdots e^{X_n\frac{t}{n}}\right)$

Due to independence of $X_1, X_2, \ldots, X_n$ $= E\left(e^{X_1\frac{t}{n}}\right) E\left(e^{X_2\frac{t}{n}}\right) \cdots E\left(e^{X_n\frac{t}{n}}\right)$

$= M_{X_1}\left(\frac{t}{n}\right) M_{X_2}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right) = \left(e^{\mu\frac{t}{n} + \frac{1}{2}\sigma^2\frac{t^2}{n^2}}\right)^n$

$= e^{\mu t + \frac{1}{2}\frac{\sigma^2}{n}t^2} \leftarrow$ MGF of Normal $\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

By uniqueness of MGF, I claim that $\bar{X} \sim Normal\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

4) a) $E(Y) = \sum_y y\,P(Y=y)$ ..

y	1	2	3	4
$P(Y=y)$	5/14	4/14	3/14	2/14

$= 1\left(\frac{5}{14}\right) + 2\left(\frac{4}{14}\right) + 3(3/14) + 4\left(\frac{2}{14}\right)$

$= \frac{5+8+9+8}{14} = \frac{30}{14} = \frac{15}{7}$

b) $E(Y|X=3) = \sum_y y\,P(Y=y|X=3) = \sum_y y\,\frac{P(Y=y, X=3)}{P(X=3)}$

$= \frac{1\cdot\left(\frac{1}{14}\right) + 2\cdot\left(\frac{1}{14}\right) + 3\cdot\left(\frac{1}{14}\right) + 4\cdot 0}{\frac{3}{14}} = \frac{\frac{6}{14}}{\frac{3}{14}} = 2$

c) $E(X|Y=3) = \sum_x x P(X=x|Y=3) = \sum_x x \frac{P(X=x, Y=3)}{P(Y=3)}$

$= \frac{1(0)+2(0)+3(\frac{1}{14})+4(\frac{1}{14})+5(\frac{1}{14})}{3/14} = \frac{\frac{12}{14}}{3/14} = 4$

5) a and b) $E(X!) = \sum_x x! P(X=x) = \sum_{x=0}^{\infty} x! \frac{e^{-\lambda}\lambda^x}{x!}$ where $\lambda > 0$ ← Poisson parameter

$= e^{-\lambda} \sum_{x=0}^{\infty} \lambda^x = e^{-\lambda} S$

$S = \sum_{x=0}^{\infty} \lambda^x = \lambda^0 + \lambda^1 + \cdots = 1 + \lambda + \lambda^2 + \cdots$ } Sum converges when $|\lambda| < 1$

$\lambda S = \lambda + \lambda^2 + \lambda^3 + \cdots$

since $\lambda > 0$, $|\lambda| = \lambda < 1$

$S - \lambda S = 1$

$S = \frac{1}{1-\lambda}$

Finally, $E(X!) = e^{-\lambda} S = \underbrace{\frac{e^{-\lambda}}{1-\lambda}}_{\text{part b)}}$ where $\underbrace{0 < \lambda < 1}_{\text{part a)}}$