Sample Questions: Continuous Random Variables

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1. The continuous random variable X has density $f(x) = \begin{cases} \frac{c}{x^{\alpha+1}} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$

   where $\alpha > 0$.

   (a) Find the constant c

   (b) Find the cumulative distribution function $F(x)$.

   (c) The median of this distribution is that point m for which $P(X \leq m) = \frac{1}{2}$. What is the median? The answer is a function of α.

2. Let $F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^\theta & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$

(a) If $\theta = 3$, what is $P\left(\frac{1}{2} < X \leq 4\right)$? The answer is a number.

(b) Find $f(x)$.

3. If a random variable has density $f(x) = \frac{1}{2}e^{-|x|}$, find the cumulative distribution function.

4. The Uniform(a, b) distribution has density $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$

Give the cumulative distribution function.

5. The Exponential(λ) distribution has density $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

(a) Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(b) Find $F(x)$

(c) Still for the exponential density with $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$, prove the "memoryless" property:

$$P(X > t + s | X > s) = P(X > t)$$

for $t > 0$ and $s > 0$. For example, the probability that the conversation lasts at least t more minutes is the same as the probability of it lasting at least t minutes in the first place.

6. The Gamma(α, λ) distribution has density $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} \, x^{\alpha-1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

   Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

7. The Normal(μ, σ) distribution has density $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$
Let $X \sim \mathrm{N}(\mu, \sigma)$ and $Z = \frac{X-\mu}{\sigma}$. Find the density of Z.

8. Let $Z \sim N(0,1)$ (standard normal), so that $f_x(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. If $x > 0$, show $F_z(-x) = 1 - F_z(x)$.

9. Let $X \sim N(\mu = 50, \sigma = 10)$.

(a) Find $P(X < 60)$. The answer is a number.

(b) Find $P(X > 30)$. The answer is a number.

(c) Find $P(30 < X < 55)$.

10. The beta density with parameters α and β is $f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$

Let $X \sim \text{Beta}(\alpha, \beta)$ with $\beta = 1$.

(a) Write the density of X for $0 \le x \le 1$. Simplify. You will prove $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ in homework.

(b) Let $Y = 1/X$. For what values of y is $f_y(y) > 0$? Show some work.

(c) Derive $f_y(y)$. Don't forget to specify where the density is greater than zero.

11. Let $Z \sim N(0,1)$ and $Y = Z^2$.

(a) For what values of y is $f_y(y) > 0$?

(b) Show that Y has a gamma distribution and give the parameters. You may use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, without proof.

12. In this problem, the random variable X is transformed by its own distribution function. Let the continuous random vabriale X have distribution function $F_x(x)$, and let $Y = F_x(X)$.

    (a) For what values of y is $f_y(y) > 0$? Hint: as x ranges from $-\infty$ to ∞, $F_x(x)$ ranges from ____ to ____.

    (b) Find $f_y(y)$.