Mixed Distributions

Our text discusses random variables that are either discrete or continuous. We will go further, and consider *mixed* random variables that have a distrete part and a continuous part. To justify this, consider observing a lightbulb until it fails. What if there is a positive probability that the failure time is zero (the bulb never goes on)?

Let X_1 be a discrete random variable, and let X_2 be (absolutely) continuous. You may think of a mixed random variable Y as arising from a two-step statistical experiment, like this. First, toss a coin with probability of a Head equal to α. If the coin shows Heads, $Y = X_1$; if it is Tails, $Y = X_2$. Denoting by C the outcome of the coin toss, we can write the distribution function of Y as

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(Y \leq y|C = h)\, P(C = h) + P(Y \leq y|C = t)\, P(C = t) \\ &= \alpha P(X_1 \leq y) + (1-\alpha) P(X_2 \leq y) \\ &= \alpha F_{X_1}(y) + (1-\alpha) F_{X_2}(y) \end{aligned}$$

Let $g(\cdot)$ be a function for which all the relevant expectations exist. By the double expectation formula (which is actually part of the *definition* of conditional probability in more advanced courses), we have

$$\begin{aligned} E[g(Y)] &= E[E[g(Y)|C]] = [E[g(Y)|C = h]\, P(C = h) + [E[g(Y)|C = t]\, P(C = t) \\ &= \alpha E[g(X_1)] + (1-\alpha) E[g(X_2)] \\ &= \alpha \sum_x g(x)\, f_{X_1}(x) + (1-\alpha) \int_{-\infty}^{\infty} g(x)\, f_{X_2}(x)\, dx \end{aligned}$$

This formula completely determines the distribution of Y, since the function g could be an indicator for any set of interest. We can even use it to *define* some notation that might otherwise be confusing. Let us write

$$\begin{aligned} E[g(Y)] &= \int g(y)\, dF_Y(y) = \int g(y)\, dP_Y(y) \\ &= \alpha \sum_x g(x)\, f_{X_1}(x) + (1-\alpha) \int_{-\infty}^{\infty} g(x)\, f_{X_2}(x)\, dx \end{aligned}$$

You will prove in homework that this "integral" enjoys all the usual properties of sums and integrals. If you later learn that it is a special case of a Lebesgue integral, no difficulty will arise. In the meantime, you will have a concrete meaning for a notation that is frequently used without much explanation.