Homework 9: Quiz Nov. 18

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common moment-generating function $M(t)$. Let $Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$. Prove $Z_n \stackrel{d}{\to} Z$, where Z is standard normal.

   Please do this problem neatly, and hand it in for 4 points out of 10. To repeat, this is a take-home question. I expect you to look the proof up in a textbook, but I want to be convinced that you have gone through the details yourself. Proofs in books always omit details. I want you to fill them in. If you are comfortable with complex variables, do the problem with characteristic functions instead of moment-generating functions.

2. Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables and let a be a real constant. Using the definitions, show

   (a) $X_n \stackrel{a.s.}{\to} X \Rightarrow aX_n \stackrel{a.s.}{\to} aX$

   (b) $X_n \stackrel{P}{\to} X \Rightarrow aX_n \stackrel{P}{\to} aX$

   (c) $X_n \stackrel{d}{\to} X \Rightarrow aX_n \stackrel{d}{\to} aX$

   You may use facts about limits of real sequences without proof.

3. Let a be a real constant. Using definitions, show $X_n \stackrel{d}{\to} a \Rightarrow X_n \stackrel{P}{\to} a$

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common distribution function $F(x)$. Define the *empirical distribution function* as $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$.

   (a) Agree or disagree: $\hat{F}_n(x)$ is the proportion of observations less than or equal to x.

   (b) Find a more convenient expression for $\int x^k d\hat{F}_n(x)$.

   (c) Agree or disagree: For each fixed x_0, $\hat{F}_n(x_0)$ is a random variable, so that $\hat{F}_n(x)$ is a *random function*.

   (d) For fixed x, can the random variable $\hat{F}_n(x)$ be continuous if $F(x)$ is a step function? What if $F(x)$ is continuous?

   (e) What are the possible values that can be assumed by $\hat{F}_n(x)$?

   (f) For fixed x, what is the *pmf* of the random variable $\hat{F}_n(x)$?

(g) For fixed x, what is the *cdf* of $\hat{F}_n(x)$?

(h) For fixed x, what is the expected value of $\hat{F}_n(x)$? Is $\hat{F}_n(x)$ unbiased for $F(x)$?

(i) For fixed x, what is the variance of $\hat{F}_n(x)$?

(j) Use the last two results to show $\hat{F}_n(x) \xrightarrow{P} F(x)$.

(k) Use moment-generating functions to show $\hat{F}_n(x) \xrightarrow{P} F(x)$.

(l) Use the Strong Law of Large Numbers to show $\hat{F}_n(x) \xrightarrow{a.s.} F(x)$.

(m) Prove that if $F(x)$ is continuous, the almost sure convergence of $\hat{F}_n(x)$ to $F(x)$ is uniform in x. This requires more work than the other parts of this question.

(n) Give a more convenient expression for the *empirical moment-generating function* $\hat{M}_n(t) = \int e^{xt} d\hat{F}_n(x)$.

(o) What is the relationship between existence of $M(t)$ and existence of $\hat{M}_n(t)$?

(p) Show $\hat{M}_n(t) \xrightarrow{a.s.} M(t)$. Under what conditions does the convergence *not* hold?