Homework 11: Quiz Dec. 2

You may bring the convergence handout.

1. Show that a collection of i.i.d. random variables with finite mean and variance satisfies the Lindeberg condition.

2. Regression through the origin: Let $Y_i = \beta x_i + e_i$, $i = 1, \ldots, n$, where x_i values are known fixed constants, β is an unknown constant, and the e_i values are i.i.d. random variables with expected value zero and variance σ^2. A *least squares estimate* of some parameter θ is defined as follows. Suppose $E[Y_i] = g_i(\theta)$. The least squares estimate is the value of θ that gets the Y_i values as close to their expected values as possible, in the sense of minimizing $Q = \sum_{i=1}^{n}(Y_i - g_i(\theta))^2$. Find the least-squares estimate of β. Under what conditions does it have a limiting normal distribution? Show your work.

3. Show that a sufficient condition for $X_n \stackrel{a.s.}{\to} X$ is $\forall \epsilon > 0$, $\sum_{k=1}^{\infty} P\{|X_k - X| \geq \epsilon\} < \infty$. Hint: Do not be tempted by Markov's inequality, because the expected values might not exist. Start with the second item on the convergence handout.

4. Let $P(X_n = n^2) = \frac{1}{n^2}$, and $P(X_n = 0) = 1 - \frac{1}{n^2}$.

   (a) Is it clear that $X_n \stackrel{P}{\to} 0$?

   (b) Does $X_n \stackrel{a.s.}{\to} 0$? Why or why not?

5. Suppose $X_1, X_2, \ldots, X_n$ are continuous uniform random variables on the interval $(0, \theta)$. Let $T_n = 2\overline{X}_n$, and let Y_n be the maximum of $X_1, X_2, \ldots, X_n$. Last week you showed both T_n and Y_n are consistent.

   (a) Why can we say $T_n - \theta = O_p(n^{-\frac{1}{2}})$?

   (b) Find the limiting distribution of $n(\theta - Y_n)$. Just find the distribution function explicitly and take the limit.

   (c) Why does this establish that Y_n is going to θ *faster* than T_n?

   (d) Clearly $T_n \stackrel{a.s.}{\to} \theta$. Prove $Y_n \stackrel{a.s.}{\to} \theta$ as well. Hint: One approach may be found elsewhere on this assignment, but perhaps it is more satisfying to use the seond item on the convergence handout directly.

6. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with expected value μ and variance σ^2.

   (a) Show $e^{\overline{X}_n}$ is asymptotically normal.

   (b) Let $f_\mu(x)$ be the density of a normal with mean μ and variance 1. Show that with suitable scaling, $f_\mu(\overline{X}_n)$ is asymptotically chi-square.

   (c) Let $F_\mu(x)$ be the cdf of a normal with mean μ and 1. Show that $F_\mu(\overline{X}_n)$ is asymptotically normal. Take an extra term in the Taylor expansion to see that it happens faster than usual.

7. This is a repeat from last week, with a better hint. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with finite fourth moment; denote their commom expected value by μ and their variance by σ^2. Show that $\sqrt{n}(S_n^2 - \sigma^2)$ (where S_n^2 is the common sample variance) converges in distribution to a normal random variable. Hint: Start by letting $Y_i = (X_i - \mu)^2$; later, show $\sqrt{n}(\overline{Y}_n - \sigma^2) - \sqrt{n}(\frac{n-1}{n} S_n^2 - \sigma^2) \xrightarrow{P} 0$. Simplify, then complete the square. You're almost done.

8. Adapt an example from lecture to show that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ need not imply $X_n + Y_n \xrightarrow{d} X + Y$, even though $g(x, y) = x + y$ is continuous. Why does this not contradict the Slutsky theorem?