STA2112F99 ε–δ Review

1. Sequences of real numbers

Definition: Let a_1, a_2, ... be a sequence of real numbers. We will write $a_n \to a$, or $\lim_{n\to\infty} a_n = a$, if for all $\varepsilon > 0$, there exists a real number N such that if $n > N$, $|a_n - a| < \varepsilon$.

For short, we can write: $a_n \to a$ means $\forall\ \varepsilon>0, \exists\ N\in \mathbb{R} \ni$ if $n > N$, $|a_n - a| < \varepsilon$.

Example 1: Show $\frac{1}{n} \to 0$.

Proof: We seek to show that $\forall\ \varepsilon>0, \exists\ N\in \mathbb{R} \ni$ if $n > N$, $|\frac{1}{n} - 0| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon$. Let $N = 1/\varepsilon$ and let $n > N$. Then $n > 1/\varepsilon \Rightarrow \frac{1}{n} < \varepsilon$, which was to be shown. ♦

Example 2: Show $\frac{n+1}{2n+6} \to \frac{1}{2}$.

Proof. We seek to show that $\forall\ \varepsilon>0, \exists\ N\in \mathbb{R} \ni$ if $n > N$, $|\frac{n+1}{2n+6} - \frac{1}{2}| < \varepsilon$. Now $|\frac{n+1}{2n+6} - \frac{1}{2}| =$ $\left|\frac{2(n+1)-(2n+6)}{2(2n+6)}\right| = \left|\frac{2n+2-2n-6)}{4(n+3)}\right| = \left|\frac{-4}{4(n+3)}\right| = \frac{1}{n+3} < \varepsilon \Leftrightarrow n+3 > 1/\varepsilon \Leftrightarrow n > 1/\varepsilon - 3$. Accordingly, let $N > 1/\varepsilon$, and the result follows. ♦

Example 3: Let $a_n \to a$ and $b_n \to b$. Show $a_n + b_n \to a + b$.

Proof 1: We seek to show that $\forall\ \varepsilon>0, \exists\ N\in \mathbb{R} \ni$ if $n > N$, $|(a_n + b_n) - (a + b)| < \varepsilon$. Let $\varepsilon>0$ be given. $a_n \to a$ implies $\exists\ N_1\in \mathbb{R} \ni$ if $n > N_1$, $|a_n - a| < \varepsilon/2$. Similarly, $b_n \to b$ implies $\exists\ N_2\in \mathbb{R} \ni$ if $n > N_2$, $|b_n - b| < \varepsilon/2$. Let $N = \mathrm{Max}(N_1, N_2)$ and let $n > N$. Then $|a_n - a| < \varepsilon/2$ *and* $|b_n - b| < \varepsilon/2$. Consequently, $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. ♦

To see more clearly what's going on here, we could do it this way. Proof 1 is more professional.

Proof 2: We seek to show that $\forall\ \varepsilon>0$, $\exists\ N\in \mathbb{R} \ni$ if $n > N$, $|(a_n + b_n) - (a + b)| < \varepsilon$. Let $\varepsilon>0$ be given.

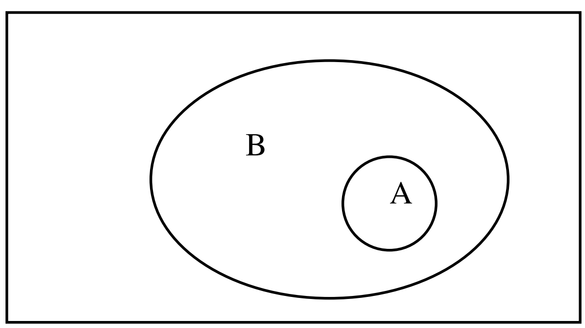
Step	Justification
1. $a_n \to a$ implies $\exists\ N_1 \in \mathbb{R} \ni$ if $n > N_1$, $|a_n - a| < \varepsilon /2$	Definition of limit.
2. $b_n \to b$ implies $\exists\ N_2 \in \mathbb{R} \ni$ if $n > N_2$, $|b_n - b| < \varepsilon /2$	Definition of limit.
3. Let $N = \mathrm{Max}(N_1, N_2)$ and let $n > N$. Then $|a_n - a| < \varepsilon /2$ *and* $|b_n - b| < \varepsilon /2$.	Using steps 1 and 2.
4. $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|$	Algebra
5. $\leq |(a_n - a| + |b_n - b|$	Triangle inequality
6. $< \varepsilon /2 + \varepsilon /2$	Step 3
7. $= \varepsilon$	Algebra
8. Done. We have found N such that for any $\varepsilon>0$, if $n > N$, $|(a_n + b_n) - (a + b)| < \varepsilon$	Satisfies definition of $a_n + b_n \to a + b$.

Example 4: Show that if $\lim_{n\to\infty} a_n$ exists, it is unique.

Proof: Let $a_n \to x$ and $a_n \to y$. We seek to show that $x = y$. Suppose on the contrary that $x \neq y$; without loss of generality, let $x > y$. Now $a_n \to x$ implies $\exists\ N_1 \in \mathbb{R} \ni$ if $n > N_1$, $|a_n - x| < (x-y)/3$. Similarly, $a_n \to y$ implies $\exists\ N_2 \in \mathbb{R} \ni$ if $n > N_2$, $|a_n - y| < (x-y)/3$. Let $N = \text{Max}(N_1, N_2)$ and let $n > N$, so that $a_n > x - (x-y)/3$ and at the same time, $a_n < y + (x-y)/3$. This leads to the conclusion that $x < y$, which contradicts our initial assumption that $x > y$. A similar argument shows that $y < x$ implies $y > x$. Therefor $x \neq y$ is impossible, so that $x = y$ as required. ♦

The idea here is that if we start out with an assumption, and we reason *correctly* to a conclusion that is obviously false, the initial assumption must have been wrong. This is called a proof by contradiction. The key is to reason correctly, because any mistake can lead to a contradiction, and that proves nothing.

Here is a Venn Diagram that may help. The diagram illustrates $A \Rightarrow B$. It also illustrates (Not B) $\Rightarrow$ (Not A). In fact, these two statements are equivalent. The idea behind proof by contradiction is to start with the assumption (Not B), and prove (Not A). This allows us to conclude $A \Rightarrow B$.



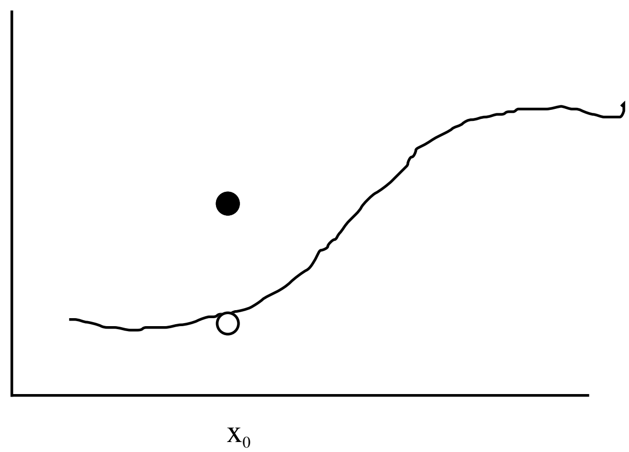
In the proof of Example 4, a couple of steps were left out. How do we know that

a. $a_n > x - (x-y)/3$ (and $a_n < y - (x-y)/3$)?

b. Even assuming (a), how does it follow that $x < y$?

2. Limits of real functions

We want to be able to define $\lim_{x \to x_0} f(x) = y$. Want f(x) to be very close to y whenever x is close to x_0, but we specifically want to leave out the case $x=x_0$. Why leave it out? Because of situations like this:



And also, consider the definition of a derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} q(x)$$

The function q(x) is not even defined at x=a.

Definition: $\lim_{x \to x_0} f(x) = y$ means $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0 \ni$ if $|x-x_0|<\delta$ with $x \neq x_0$ then $|f(x)-y| < \varepsilon$.

Example 5: Let f(x) = 3x + 1. Show $\lim_{x \to 2} f(x) = 7$.

Proof: We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0 \ni$ if $|x-2|<\delta$ with $x \neq 2$, then $|(3x+1) - 7| < \varepsilon$. Now $|3x-6| < \varepsilon \Leftrightarrow 3\ |x-2| < \varepsilon \Leftrightarrow |x-2| < \varepsilon/3$. Set $\delta = \varepsilon/3$, and the result follows. ♦

Notice we never needed to use the assumption that $x \neq 2$; that's because in this case, $|x-x_0|<\delta \Rightarrow |f(x)-y| < \varepsilon$ even when $x=x_0$. Also notice that we can make the proof more straightforward by putting it in tabular form and proceeding directly from premise to conclusion.

Step	Justification
1. We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0$ $\ni$ if $\lvert x-2\rvert<\delta$ with $x\neq 2$, then $\lvert(3x+1) - 7\rvert < \varepsilon$.	Definition of $\lim_{x\to 2} f(x) = 7$
2. Set $\delta = \varepsilon/3$;	This choice of δ will satisfy the definition.
3. $\lvert x-2\rvert < \varepsilon/3$	Substitute $\varepsilon/3$ for δ
4. $\Rightarrow\ 3\lvert x-2\rvert < \varepsilon$	Algebra
5. $\Rightarrow\ \lvert 3x-6\rvert < \varepsilon$	Algebra
6. $\Rightarrow\ \lvert(3x+1) - 7\rvert < \varepsilon$	Algebra
7. Done.	Definition of $\lim_{x\to 2} f(x) = 7$ is satisfied.

This is all we were doing in the compact proof above, just following the arrows backwards.

Example 6: Let $f(x) = x^2$ if $x\neq 0$, and $f(x)=4$ if $x=0$. Prove that $\lim_{x\to 0} f(x) = 0$.

Proof: We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0 \ni$ if $\lvert x-0\rvert<\delta$ with $x\neq 0$, then $\lvert f(x)-0\rvert < \varepsilon$. Assume $x\neq 0$ and let $\varepsilon>0$ be given. Then $\lvert f(x)\rvert = \lvert x^2\rvert = x^2 < \varepsilon \Leftrightarrow \lvert x\rvert < \sqrt{\varepsilon}$. Set $\delta = \sqrt{\varepsilon}$, and the result follows. ♦

Example 7: Let $f(x) = \dfrac{7x^2 + 11x - 6}{x + 2}$. Show $\lim_{x\to 0} f(x) = -3$.

Proof 1: We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta>0 \ni$ if $\lvert x-0\rvert<\delta$ with $x\neq 0$, then $\left|\dfrac{7x^2 + 11x - 6}{x + 2} - (-3)\right| < \varepsilon$. Assume $x\neq 0$ and let $\varepsilon>0$ be given. Then $\left|\dfrac{7x^2 + 11x - 6}{x + 2} - (-3)\right| =$
$\left|\dfrac{7x^2 + 11x - 6}{x + 2} + 3\right| = \left|\dfrac{7x^2 + 11x - 6}{x + 2} + \dfrac{3(x + 2)}{x + 2}\right| = \left|\dfrac{7x^2 + 11x - 6 + 3x + 6}{x + 2}\right| = \left|\dfrac{7x^2 + 14x}{x + 2}\right| =$
$\left|\dfrac{7x(x + 2)}{x + 2}\right| = \lvert 7x\rvert = 7\lvert x\rvert < \varepsilon \Leftrightarrow \lvert x\rvert < \varepsilon/7$. Accordingly, let $\delta = \varepsilon/7$, and the result follows. ♦

You may still be at the stage of "Huh? What do you mean it follows?" The problem may be that we were working backwards from what we sought to show, and obtained an equivalent statement of the form $|x-x_0|<\delta$ with $x\neq x_0$ (though we did not need $x\neq x_0$). Okay, here is a proof that is more "obvious" in the sense that it starts with $|x-x_0|<\delta$ and proceeds to $|f(x)-y| < \varepsilon$. But it's less obvious because it's a mystery where the δ came from. Well, it came from the backwards proof above.

Proof 2: We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0 \ni$ if $|x-0|<\delta$ with $x\neq 0$, then $\left|\frac{7x^2+11x-6}{x+2}-(-3)\right| < \varepsilon$. Let $\varepsilon>0$ be given, let $\delta = \varepsilon/7$, and let $|x|<\delta$ with $x\neq x_0$. Then $|x| < \varepsilon/7 \Rightarrow$ $7|x| < \varepsilon \Rightarrow \left|\frac{7x(x+2)}{x+2}\right| < \varepsilon \Rightarrow \left|\frac{7x^2+14x}{x+2}\right| < \varepsilon \Rightarrow \left|\frac{7x^2+11x-6+3x+6}{x+2}\right| < \varepsilon \Rightarrow$ $\left|\frac{7x^2+11x-6}{x+2}+\frac{3(x+2)}{x+2}\right| < \varepsilon \Rightarrow \left|\frac{7x^2+11x-6}{x+2}+3\right| < \varepsilon \Rightarrow \left|\frac{7x^2+11x-6}{x+2}-(-3)\right| < \varepsilon$. This is what we needed to show. ♦

What is meant by "the result follows" is exactly Proof 2; that is, Proof 2 is implicit in Proof 1. Maybe Proof 2 is easier to follow in a way, but I think Proof 1 is better, because the thought process behind it is clearer. And in order to write Proof 2, you need to have done Proof 1 first anyway.

Example 8: Assume $\lim_{x\to x_0} f(x) = 3$. Show $\lim_{x\to x_0} 2\,f(x) = 6$.

Proof: We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0 \ni$ if $|x-x_0|<\delta$ with $x\neq x_0$ then $|2f(x)-6| < \varepsilon$. Let $\varepsilon>0$ be given. $\lim_{x\to x_0} f(x) = 3$ means $\exists\ \delta_1>0 \ni$ if $|x-x_0|<\delta_1$ with $x\neq x_0$ then $|f(x)-3| < \varepsilon/2$. Choose $\delta = \delta_1$, and let $|x-x_0|<\delta$ with $x\neq x_0$. Then $|f(x)-3| < \varepsilon/2 \Rightarrow |2f(x)-6| < \varepsilon$, which was to be shown. ♦

Example 9: Assume $\lim_{x\to x_0} f(x) = y_1$ and $\lim_{x\to x_0} g(x) = y_2$. Prove $\lim_{x\to x_0} [f(x) + g(x)] = y_1 + y_2$.

Proof: We seek to show that $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon,x_0)>0 \ni$ if $|x-x_0|<\delta$ with $x\neq x_0$ then $|f(x)+g(x)-(y_1+y_2)| < \varepsilon$. Let $\varepsilon>0$ be given. $\lim_{x\to x_0} f(x) = y_1$ implies $\exists\ \delta_1 \ni$ if $|x-x_0|<\delta_1$ with $x\neq x_0$ then $|f(x)-y_1| < \varepsilon/2$. Similarly, $\lim_{x\to x_0} g(x) = y_2$ implies $\exists\ \delta_2 \ni$ if $|x-x_0|<\delta_2$ with $x\neq x_0$ then $|g(x)-y_2| < \varepsilon/2$. Take $\delta = \text{Min}(\delta_1,\delta_2)$ and let $|x-x_0|<\delta$ with $x\neq x_0$. Then $|f(x)-y_1| < \varepsilon/2$ *and*

$|f(x)-y_2| < \varepsilon/2$. Consequently, $|f(x)+g(x)-(y_1+y_2)| = |(f(x)-y_1)+(g(x)-y_2)| \leq |f(x)-y_1| + |g(x)-y_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. ♦

Note the similarity of this proof to the one for $a_n \to a$ and $b_n \to b \Rightarrow a_n + b_n \to a + b$.

Example 10: Let $f(x) = \cos(1/x)$. Prove $\lim\limits_{x \to 0} f(x)$ does not exist.

Proof: We seek to show that there exists $\varepsilon>0$ such that there can be no pair of real numbers (y,δ) such that if $|x-0|<\delta$ with $x \neq 0$ then $|\cos(1/x)-y| < \varepsilon$. Suppose on the contrary that there is such a pair (y,δ), and let ε be given, $0< \varepsilon <1$. Note that $\exists\ x_1 \in (0,\delta)$ and $x_2 \in (0,\delta)$ with $f(x_1) = \cos(1/x_1) = 1$ and $f(x_2) = \cos(1/x_2) = -1$. Now by the definition of a limit, $|f(x_1)-y|<\varepsilon<1$ and $|f(x_2)-y|<\varepsilon<1$, so that $|f(x_1)-f(x_2)| = |f(x_1)-y + y-f(x_2)| \leq |f(x_1)-y| + |y-f(x_2)| < 2\varepsilon <2$. But $|f(x_1)-f(x_2)| = 2$. This contradiction shows that the assumed existence of the pair (y,δ) must have been false. That is, the limit does not exist. ♦

Definition: f continuous at x_0 means $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Example 11: Show $f(x) = 3x + 1$ is continuous everywhere.
Example 12: Let $f(x) = x^2$ if $x \neq 0$, and $f(x)=4$ if $x=0$. Prove that f is not continuous at $x=0$.
Example 13: Show $f(x) = \sqrt{x}$ is continuous for $x > 0$.
Example 14: Prove that if $f(x)$ and $g(x)$ are both continuous at x_0, then $f(x)+g(x)$ is also continuous at x_0.
Example 15: Prove that if $f(x)$ is continuous at x_0, then $g(x) = 3f(x)$ is also continuous at x_0.

Foundations of the Real Number System

- $\mathbb{N} = \{1,2, \ldots \}$ is the set of natural numbers.
- Define addition. $\mathbb{N}$ is closed under addition.
- Define subtraction as the inverse of addition. If a-b is allowed for b≥a, we have an

extension of $\mathbb{N}$ to the set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$.

- Define multiplication as repeated addition or subtraction. $\mathbb{Z}$ is closed under multiplication.

- Define division as the inverse of multiplication. If a = b/c is allowed for $a \notin \mathbb{Z}$ (provided $c \neq 0$) we have an extension of $\mathbb{Z}$ to the set of rational numbers $\mathbb{Q}$.

- Define an upper bound of a set of numbers A as follows. x is an upper bound of A if $\forall a \in A$, $x \geq a$.

- Define the least upper bound (LUB) of A as follows: y is a LUB of A if (1a) it is an upper bound of A, and (1b) for each upper bound x of A, $y \leq x$. An equivalent definition is: y is the LUB of A if (2a) $\forall a \in A$, $y \geq a$, and (2b) $\forall\ \varepsilon > 0, \exists\ a \in A \ni a > y - \varepsilon$.

Proof that the two definitions of LUB are equivalent: (1a) is the same as (2a) by the definition of an upper bound. It remains to show (1b) $\Leftrightarrow$ (2b).

(1b) $\Rightarrow$ (2b): Suppose (1b) holds, but there is an $\varepsilon > 0$ with no $a \in A \ni a > y - \varepsilon$. Then $y - \varepsilon$ is an upper bound of A. But since $y < y - \varepsilon$, (1b) is contradicted; (2b) follows. ♦

(2b) $\Rightarrow$ (1b): Suppose (2b) holds, but there is an upper bound x of A with $x < y$. Let $\varepsilon = (y-x)/2$ in (2b), so that $\exists\ a \in A \ni a > y - (y-x)/2 = (x+y)/2 > x$. This contradicts the assumption that x is an upper bound, so $y \leq x$ for any upper bound x. ♦

- If we add to the set of rational numbers the set of least upper bounds of all bounded subsets of rational numbers, we get another extension -- this time to the set $\mathbb{R}$ of real numbers. Another version of this is the Least Upper Bound Axiom, which says that for any set of real numbers that is bounded above, the LUB always exists.

- A real number that is not rational is called irrational. How do we know there are any irrationals?

Proposition: $\sqrt{2}$ is irrational.

Proof: Suppose on the contrary that $\sqrt{2}$ is rational. This means it can be written as a ratio a/b, where a and b are positive integers that have *no common factor* (except 1). It follows that a^2 and b^2 also have no common factors except 1. Now $a/b = \sqrt{2} \Rightarrow (a/b)^2 = 2 \Rightarrow a^2 = 2b^2$. This means b^2 is a factor of a^2. Since a^2 and b^2 have no common factors except 1, this means $b^2 = 1 \Rightarrow b=1 \Rightarrow a^2=2$. Since there is clearly no positive integer whose square equals two, the assumption that $\sqrt{2}$ is rational has led to a contradiction. Therefore, $\sqrt{2}$ is irrational.

There is a story that one of the Pythagoreans discovered that $\sqrt{2}$ is irrational (so that not all numbers were rational and a lot of important philosophic principles were thrown into question). He wouldn't keep quiet about it, so they drowned him in a lake.

- If $\{a_1, a_2, \ldots\}$ is a bounded non-decreasing set of real numbers with LUB = x, then $\lim_{n\to\infty} a_n = x$.

Proof:

- The preceding result implies that *any real number can be written uniquely as an infinite decimal*. For example, let $x \in [0,1]$. $x=0.e_1e_2e_3 \ldots$ means the following: let e_1 be the largest integer with $e_1/10 \leq x$. Let e_2 be the largest integer with $e_1/10 + e_2/100 \leq x$. In general, let $S_{n-1}=\sum_{k=1}^{n-1} \frac{e_k}{10^k}$, and define e_n to be the largest integer with $S_{n-1} + e_n/10^n \leq x$.

To see that the sequence $e_1, e_2, \ldots$ corresponds uniquely to x, it is enough to show that the sequence may be obtained from x, and x may be obtained from the sequence (so they are one-to-one). We see above how the sequence is obtained from the number x. To prove that the number x can be recovered from the sequence, note that $\{S_1, S_2, \ldots\}$ is non-decreasing, and its least upper bound is x. Therefore it converges to x, and we may obtain x from $\{e_1, e_2, \ldots\}$ by constructing $\{S_1, S_2, \ldots\}$ from $\{e_1, e_2, \ldots\}$, and then taking $\lim_{n\to\infty} S_n$.

Countable and uncountable sets

- A set B is said to be countable (denumerable) iff the elements of B can be placed in one-to-one correspondence with a subset of $\mathbb{N}$, the set of natural numbers.
- Any finite set is countable.
- The set of positive even integers is countable.
- The set of integers is countable.
- In a sense, then, there is the same "number" of positive integers, positive even integers, and integers. This is very psychedelic, man.
- Any countable union of countable sets is countable.

- The set of RATIONAL numbers is countable.
- How about the set of REAL numbers $\mathbb{R}$? Is $\mathbb{R}$ countable? This question is profoundly important to probability.

To see how essential the notion of countability is, consider the standard axioms of probability: Let S be a set, and let $\boldsymbol{\mathcal{A}}$ be a collection of subsets of S. The probability set function $P(\cdot)$ satisfies

1) $P(A) \geq 0 \;\forall\; A \in \boldsymbol{\mathcal{A}}$.
2) $P(S)=1$.
3) If $\{A_1, A_2, \ldots\}$ is a sequence of disjoint sets in $\boldsymbol{\mathcal{A}}$, then

$$P[\bigcup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} P(A_n) .$$

Recall another standard elementary definition: if X is a continuous random variable with density f, then $\forall$ real a and b with $a \leq b$, $P(a \leq X \leq b) = \int_a^b f(x)\, dx$.

Now suppose the real numbers **are** countable. Then the ones between a and b are countable too, and they can be listed in some order $x_1, x_2, \ldots$. Then $\{x: a \leq x \leq b\} = \{x_1\} \cup \{x_2\} \cup \ldots$. These singleton set are disjoint, and so

$$P[a \leq X \leq b] = P[X=x_n, \text{ some } n=1, 2, \ \ldots] = P[\bigcup_{n=1}^{\infty} \{X=x_n\}] = \sum_{n=1}^{\infty} P(X=x_n) = P[x_n \leq X \leq x_n]$$

$$\sum_{n=1}^{\infty} \int_{x_n}^{x_n} f(x)\, dx = \sum_{n=1}^{\infty} 0 = 0.$$

This shows that if the real numbers are countable, we have a disaster. Continuous probability is impossible! The real numbers had better be uncountable or we have all wasted a lot of time.

Proof that the real numbers are uncountable. It is enough to show that the numbers between zero and one are uncountable. Suppose, on the contrary, that they are countable. Then they can be listed in some order x_1, x_2, ... Write them as infinite decimals (we know from before that we can do this).

$$
\begin{aligned}
x1 &= .\, e_{11}\, e_{12}\, e_{13}\, e_{14}\, e_{15} \cdots \\
x2 &= .\, e_{21}\, e_{22}\, e_{23}\, e_{24}\, e_{25} \cdots \\
x3 &= .\, e_{31}\, e_{32}\, e_{33}\, e_{34}\, e_{35} \cdots \\
x4 &= .\, e_{41}\, e_{42}\, e_{43}\, e_{44}\, e_{45} \cdots \\
x5 &= .\, e_{51}\, e_{52}\, e_{53}\, e_{54}\, e_{55} \cdots \\
&\quad\vdots
\end{aligned}
$$

Now construct another real number $y = .\, a_1\, a_2\, a_3\, a_4\, a_5 \ldots$ as follows. For k = 1, 2, ..., if $e_{kk}=5$, let $a_k=2$. Otherwise, let $a_k=5$. Clearly y differs from x_1 in the first decimal place, y differs from x_2 in the second place, and so on. So y is not on the list. But y MUST be on the list, because the list contains all the real numbers. This contradiction shows that the construction of a list of all the real numbers is impossible. In other words, the real numbers are uncountable. Done.

- There is more than one kind of infinity, countable and uncountable. This is very cosmic.

- Not only do irrational numbers exist, but they greatly outnumber the rationals.
- Fortunately, continuous probability is possible.
- We're going to use countable and uncountable sets a lot.
- Define the *supremum* (sup) of a set of real numbers as follows. If the set is bounded above, the sup (pronounced soup) is the least upper bound. If unbounded, it's ∞. Maximum likelihood doesn't work well with the maximum. It is defined in terms of the sup.
- The *infimum* (inf) is either the greatest lower bound, or $-\infty$ if the set is not bounded below.