Bartlett corrections for likelihood ratio statistics

Assume as usual a random sample from $f(y;\theta)$, with $\theta \in R^p$; we write $\theta = (\psi, \lambda)$ with $\psi \in R^q$ the parameter of interest and λ the nuisance parameter. The likelihood ratio statistics (for simple and composite hypotheses respectively) have an asymptotic chi-square distribution:

$$\begin{aligned} w(\theta) = & \quad 2\{\ell(\hat{\theta}) - \ell(\theta)\} & \xrightarrow{d} \chi^2_p \\ w(\psi) = & \quad 2\{\ell(\hat{\psi}, \hat{\lambda}) - \ell(\psi, \hat{\lambda}_\psi)\} & \xrightarrow{d} \chi^2_q. \end{aligned}$$

This is a first order result, i.e. $\text{pr}_\theta\{w(\psi) \le w^0\} = \text{pr}\{\chi^2_q \le w^0\}\{1 + O(n^{-1})\}$.

- In i.i.d. sampling,

  $$E_\theta w(\psi) = q\{1 + b(\theta)/n + O(n^{-2})\}$$

  and thus $E_\theta w'(\psi) = q\{1 + O(n^{-2})\}$, where $w' = w/\{1 + b(\theta)/n\}$.

- It can be shown (Lawley, 1956; BN&C, 1984; McCullagh, 1987)

  $$\text{pr}_\theta\{w'(\psi) \le w^0\} = \text{pr}\{\chi^2_q \le w^0\}\{1 + O(n^{-2})\}$$

- In practice $b(\theta)$ will be replaced by $b(\hat{\theta})$. The above result still holds, even to $O(n^{-2})$ (Barndorff-Nielsen & Hall, 1988).

- An explicit expression for $b(\theta)$ is given in B-N&C, 1994, Ch.6.5.

- Bickel & Ghosh (1990) prove this result and also prove a Bayesian analogue:

  $$\text{pr}_{\pi|Y}\{w'(\psi) \le w^0\} = \text{pr}\{\chi^2_q \le w^0\}\{1 + O(n^{-2})\},$$

  where $\text{pr}_{\pi|Y}\{h(y,\theta) \le h^0\} = \int 1\{h(y,\theta) \le h^0\}\pi(\theta|y)d\theta$.