Probability and Stochastic Processes I I - Lecture 6d

Michael Evans
University of Toronto
https://utstat.utoronto.ca/mikevans/stac62/staC632024.html

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VI.4 Queueing Theory

- consider a queue of customers arriving at a server

- suppose customers arrive at the queue according to interrival times $Y_1, Y_2, \ldots \overset{i.i.d.}{\sim}$ exponential(λ) and the service times of these customers are $S_1, S_2, \ldots \overset{i.i.d.}{\sim}$ exponential(μ),

- let $T_n = \sum_{i=1}^{n} Y_i =$ arrival time of n-th customer and

$$Q_t = \# \text{ of customers in the queue at time } t$$

- with these assumptions $\{Q_t : t \geq 0\}$ is called a *M/M/1 queue*

- clearly, because of the memoryless property of the exponential distribution, for any $0 \leq t_1 \leq \cdots \leq t_n \leq t$, then

$$P(Q_t = j \mid Q_{t_i} = j_i \text{ for } i = 1, \ldots, n) = P(Q_t = j \mid Q_{t_n} = j_n)$$

and this depends on time only through $t - t_n$, so $\{Q_t : t \geq 0\}$ is a time homogeneous Markov process

- note that the number of arrivals is a Poisson process of intensity λ

Proposition VI.28 A M/M/1 queue with interarrival times exponential(λ) and service times exponential(μ) has generator matrix

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & \dots \\ \mu & -\lambda-\mu & \lambda & \\ 0 & \mu & -\lambda-\mu & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

Proof: See text and use Lemma VI.14.

Proposition VI.29 If $\lambda < \mu$, the stationary distribution π of a M/M/1 queue with interarrival times exponential(λ) and service times exponential(μ) is geometric($1-\lambda/\mu$).

Proof: The distribution π satisfies $\pi G = 0$. So $\pi_0(-\lambda) + \pi_1(\mu) = 0$ implying $\pi_1 = (\lambda/\mu)\pi_0$. Also $\pi_0(\lambda) + \pi_1(-\lambda-\mu) + \pi_2(\mu) = 0$ or $\pi_0(\lambda) + \pi_0(\lambda/\mu)(-\lambda-\mu) + \pi_2(\mu) = \pi_0(-\lambda^2/\mu) + \pi_2(\mu) = 0$ so $\pi_2 = (\lambda/\mu)^2\pi_0$ and in general $\pi_k = (\lambda/\mu)^k\pi_0$. since $1 = \sum_{k=0}^{\infty}\pi_k$ this implies that $\pi_0 = \left(\sum_{k=1}^{\infty}(\lambda/\mu)^k\right)^{-1} = 1-\lambda/\mu$ which completes the proof. ■

Proposition VI.30 If $\lambda < \mu$, then for a M/M/1 queue with interarrival times exponential(λ) and service times exponential(μ)

$$\lim_{t\to\infty} P(Q_t = k) = (\lambda/\mu)^k(1-\lambda/\mu).$$

Proof: Since the transition $i \to i+1$ always has positive probability the process is irreducible. So by Prop. VI.26 the result follows. ■

- what happens when $\lambda \geq \mu$? it can be shown that $\lim_{k\to\infty}\lim_{t\to\infty} P(Q_t \geq k) \to 1$

- when $\lambda < \mu$ and t is large there are k individuals in the queue with probability $(\lambda/\mu)^k(1-\lambda/\mu)$ and so a newly arrived individual will have to wait $W = S_1 + \cdots + S_k \sim$ gamma(k, μ) for service so (intuitively) when t is large, and $G(\cdot; k, \mu)$ is the gamma(i, μ) cdf,

$$P(W > w) = \sum_{k=0}^{\infty} (\lambda/\mu)^k(1-\lambda/\mu)(1 - G(w; k, \mu) < \infty$$

- a *G/G/1 queue* assumes interarrival times Y_n are $i.i.d.$, the service times S_n are $i.i.d.$ independent of the interarrvial times but these distributions are not assumed to be exponential

VI.5 Renewal Theory

- consider r.v.'s $Y_1, Y_2, Y_3, \ldots$ which are mutually stat. ind. and $Y_2, Y_3, \ldots$ are $i.i.d.$

- consider a machine that has been functioning properly but after Y_1 units of time from when it is being monitored, it needs maintenance and then needs maintenance after Y_2 units of time, then Y_3 units of time, etc. and denote the maintenance times times by $T_n = Y_1 + \cdots + Y_n$ with $T_0 = 0$

Definition VI.10 The process $\{N_t : t \in T\}$, where $N_t = \#\{n : T_n \leq t\} = \#$ of renewals up to time t, is called a *renewal process.*

- if $Y_1, Y_2, Y_3, \ldots$ are $i.i.d.$, then there is *zero delay* and and when the common distribution is exponential(λ), then this is a Poisson process

- fact, $\{N_t : n \in \mathbb{N}_0\}$ is a Markov process iff it is a Poisson process (recall memoryless property of exponential distribution)

Example VI.10 *Renewal process from a Markov chain*

- let $\{X_n : n \in \mathbb{N}_0\}$ be an irreducible, recurrent Markov chain and assume $X_0 = j$

- put $T_1(i) = \min\{k : X_k = i\}$ and for $n \geq 2$ put $T_n(i) = \min\{k > T_{n-1}(i) : X_k = i\}$ = time of n-th visit to state i

- put $Y_n = T_n - T_{n-1}$ so

$$P(Y_1 = y \,|\, X_0 = j) = P(X_y = i, X_m \neq i, 1 \leq m \leq y-1 \,|\, X_0 = j)$$

and using MP, for $n \geq 2$

$$\begin{aligned} & P(Y_n = y \,|\, X_{n-1} = i) \\ = \; & P(X_y = i, X_m \neq i, 1 \leq m \leq y-1 \,|\, X_0 = i) \text{ by TH} \\ = \; & P(Y_1 = y \,|\, X_0 = i) \end{aligned}$$

- note for $z > y$

$$\begin{aligned} & P(Y_1 = y, Y_2 = z \,|\, X_0 = j) \\ = \; & P(Y_2 = z \,|\, Y_1 = y, X_0 = j) P(Y_1 = y \,|\, X_0 = j) \text{ and} \end{aligned}$$

$$
\begin{aligned}
& P(Y_2 = z \,|\, Y_1 = y, X_0 = j) \\
= & \begin{array}{l} P(X_{y+z} = i, X_m \neq i, y+1 \leq m \leq z+y-1 \,| \\ X_y = i,\ X_m \neq i, 1 \leq m \leq y-1, X_0 = j) \end{array} \\
= & P(X_{y+z} = i, X_m \neq i, y+1 \leq m \leq z+y-1 \,|\, X_y = i) \text{ by MP} \\
= & P(X_z = i, X_m \neq i, 1 \leq m \leq z-1 \,|\, X_0 = i) \text{ by TH} \\
= & P(Y_1 = z \,|\, X_0 = i)
\end{aligned}
$$

so in general $Y_1, Y_2, Y_3, \ldots$ are mutually stat. ind. and $Y_2, Y_3, \ldots$ are $i.i.d.$ $P(Y_i = z \,|\, X_0 = i) = P(Y_1 = z \,|\, X_0 = i)$
■

- consider N_t/t = renewal rate

Proposition VI.31 *(Elementary Renewal Theorem) For a renewal process* $\{N_t : t \in T\}$ with $E(Y_n) = \mu < \infty$ *for* $n \geq 2$, then as $t \to \infty$

(i) $\frac{N_t}{t} \overset{wp1}{\to} \frac{1}{\mu}$, (ii) $E\left(\frac{N_t}{t}\right) \to \frac{1}{\mu}$.

Proof: (i) By the SLLN $\frac{1}{n}T_n = \frac{1}{n}\sum_{i=1}^n Y_i \overset{wp1}{\to} \mu$ which implies $\lim_{t\to\infty} N_t = \lim_{t\to\infty} \#\{n : T_n \leq t\} = \infty$ wp1 (otherwise $T_n = \infty$ infinitely often which contradicts $\mu < \infty$). Therefore,

$$\lim_{t\to\infty} \frac{T_{N_t}}{N_t} = \lim_{n\to\infty} \frac{T_n}{n} = \mu \text{ wp1}$$

and since

$$\frac{T_{N_t}}{N_t} \leq \frac{t}{N_t} \leq \frac{T_{N_t+1}}{N_t}$$

this implies $t/N_t \overset{wp1}{\to} \mu$.

(ii) fact. ■

Example VI.10 *Renewal process from a Markov chain (continued)*

- recall $T_n(i) = \min\{k > T_n(i) : X_k = i\}$ = time of n-th visit to state i

- when the chain is positive recurrent then $\mu = m_i$ mean recurrence time $< \infty$ and recall the unique stationary distribution is given by $\pi_i = 1/m_i$ ■

Proposition VI.32 *(Blackwell Renewal Theorem)* For a renewal process with $\mu < \infty$ and s.t. Y_2 is non-arithmetic (there isn't $\delta > 0$ s.t. $P(Y_2 \in \{i\delta : i \in \mathbb{Z}\}) = 1$) then $\lim_{t\to\infty} E(N_{t+h} - N_t) = h/\mu$ for any $h > 0$.

Proof: fact

Corollary VI.33 If $h > 0$ is s.t. $P(Y_2 < h) = 0$, then $\lim_{t\to\infty} P(\exists n \text{ s.t. } t < T_n \leq t + h) = h/\mu$.

Proof: We have

$$\begin{aligned} & P(\exists n \text{ s.t. } t < T_n \leq t + h) = P(N_{t+h} - N_t \geq 1) \\ = \; & P(N_{t+h} - N_t = 1) = E(N_{t+h} - N_t) \end{aligned}$$

whence the result follows from the proposition. ■

- suppose at renewal i there is a reward (or cost) R_i where $R_1, R_2, \ldots$ are $i.i.d.$ and define the *renewal reward process* by

$$R_t = \sum_{i=1}^{N_t} R_i = \text{ total reward up to time } t$$

Proposition VI.34 *(Renewal Reward Theorem)* For a renewal process with $\mu < \infty$, then $\frac{R_t}{t} \stackrel{wp1}{\to} \frac{E(R_1)}{\mu}$.

Proof: We have

$$\frac{R_t}{t} = \frac{1}{t} \sum_{i=1}^{N_t} R_i = \frac{N_t}{t} \left(\frac{1}{N_t} \sum_{i=1}^{N_t} R_i \right)$$

and by the SLLN

$$\lim_{t \to \infty} \frac{1}{N_t} \sum_{i=1}^{N_t} R_i = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} R_i = E(R_1).$$

■

Exercise VI.14 Text 4.5.4

Exercise VI.15 Text 4.5.6

Exercise VI.16 Text 4.6.9

Exercise VI.17 Text 4.6.10