Probability and Stochastic Processes I I - Lecture 6b

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2024

VI.2 Poisson Process

Definition VI.5 A process $\{N_t : t \geq 0\}$ is a (homogenous) *Poisson process* of *intensity* $\lambda > 0$ if
(i) $N_0 = 0$ (ii) if $0 \leq t_1 < \cdots < t_n$, then $N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent and $N_{t_i} - N_{t_{i-1}} \sim \text{Poisson}(\lambda(t_i - t_{i-1}))$.

- note - $N_t = N_t - N_0 \sim \text{Poisson}(\lambda t)$ so $P(N_t = j) = (\lambda t)^j e^{-\lambda t}/j!$ for $j = 0, 1, 2, \ldots$

- $N_t =$ a count of something occurring in $[0, t]$ as, when $s < t, N_t = N_t - N_s + N_s \geq N_s$ with probability 1

- recall when $X \sim \text{Poisson}(\lambda)$ then $E(X) = Var(X) = \lambda$ and mgf $m_X(t) = \exp(\lambda(e^t - 1))$

- recall when $X_1 \sim \text{Poisson}(\lambda_1)$ stat. ind. of $X_2 \sim \text{Poisson}(\lambda_2)$, then $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

- also, if $X_n \sim \text{binomial}(n, p_n)$ where $p_n = \lambda/n + o(n)$ then $X_n \xrightarrow{d} \text{Poisson}(\lambda)$ as $n \to \infty$ (see Lecture 22 for STAC62)

Proposition VI.12 A Poisson process $\{N_t : t \geq 0\}$ is a Markov process and $\{N_t - \lambda t : t \geq 0\}$ is a martingale.

Proof: Suppose $0 < t_1 < \cdots < t_n$. Then, for $j_n \geq j_{n-1} \geq \cdots \geq j_1 \geq 0$

$$\begin{aligned}
& P(N_{t_n} = j_n \,|\, N_{t_1} = j_1, \ldots, N_{t_{n-1}} = j_{n-1}) \\
= \;& P(N_{t_n} - N_{t_{n-1}} + N_{t_{n-1}} = j_n \,|\, N_{t_1} = j_1, \ldots, , N_{t_{n-1}} = j_{n-1}) \\
= \;& P(N_{t_n} - N_{t_{n-1}} = j_n - j_{n-1}) \text{ by independent increments} \\
= \;& P(N_{t_n} - N_{t_{n-1}} = j_n - j_{n-1} \,|\, N_{t_{n-1}} = j_{n-1}) \text{ by ind. increments} \\
= \;& P(N_{t_n} = j_n \,|\, N_{t_{n-1}} = j_{n-1}).
\end{aligned}$$

Also,

$$\begin{aligned}
& E(N_{t_n} - \lambda t_n \,|\, N_{t_1} = j_1, \ldots, , N_{t_{n-1}} = j_{n-1}) \\
= \;& E(N_{t_n} - \lambda t_n \,|\, N_{t_{n-1}} = j_{n-1}) \text{ because a MP} \\
= \;& E(N_{t_n} - N_{t_{n-1}} - \lambda t_n + N_{t_{n-1}} \,|\, N_{t_{n-1}} = j_{n-1}) \\
= \;& E(N_{t_n} - N_{t_{n-1}}) - \lambda t_n + E(N_{t_{n-1}} \,|\, N_{t_{n-1}} = j_{n-1}) \text{ ind. increments} \\
= \;& \lambda(t_n - t_{n-1}) - \lambda t_n + j_{n-1} = N_{t_{n-1}} - \lambda t_{n-1}.
\end{aligned}$$

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- recall $X \sim \text{exponential}_{rate}(\lambda)$ has density $f_X(x) = \lambda e^{-\lambda x}$ for all $x \geq 0$ ($\text{exponential}_{scale}(\lambda)$ has density $\lambda^{-1}e^{-x/\lambda}$)

- has cdf $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$, $E(X) = 1/\lambda$, $Var(X) = 1/\lambda^2$, mgf $m_X(t) = \lambda/(\lambda - t)$ for $t < \lambda$

- the $\text{gamma}_{rate}(\alpha, \lambda)$ distribution has density

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1-} e^{-\lambda x} \text{ for } x \geq 0$$

$E(X) = \alpha/\lambda$, $Var(X) = \alpha/\lambda^2$ and mgf $m_X(t) = \lambda^\alpha(\lambda - t)^{-\alpha}$ for $t < \lambda$

- so $\text{exponential}_{rate}(\lambda) = \text{gamma}_{rate}(1, \lambda)$

- if $X_1 \sim \text{gamma}_{rate}(\alpha_1, \lambda)$ stat. ind. of $X_2 \sim \text{gamma}_{rate}(\alpha_2, \lambda)$ then $X_1 + X_2 \sim \text{gamma}_{rate}(\alpha_1 + \alpha_2, \lambda)$

- fact: a Poisson process satisfies the strong Markov property: if T is a finite stopping time for $\{N_t : t \geq 0\}$ then $\{N_{T+t} - N_T : t \geq 0\}$ is a Poisson process of intensity λ which is independent of $\{N_s : s \leq T\}$

- let $T_1 = \inf\{t : N_t > 0\}$ then $\{T_1 \leq t\} \in \mathcal{A}_{\{N_s : 0 \leq s \leq t\}}$ so it is a stopping time for the process

- let $T_i = \inf\{t : N_{T_{i-1}+t} - N_{T_{i-1}} > 0\}$ for $i \geq 1$

- $T_1, T_2, \ldots$ are called the *interarrival times*

Proposition VI.13 $T_1, T_2, \ldots \overset{i.i.d.}{\sim}$ exponential(λ).

Proof: We have

$$P(T_1 > t) = P(N_t = 0) = e^{-\lambda t}$$

which is 1 - cdf of T_1 and so $T_1 \sim$ exponential(λ). Then $\{T_2 > t\} = \{N_{T_1+t} - N_{T_1} = 0\}$ which, by the SMP is independent $\{N_s : s \leq T_1\}$, and so T_2 is independent of T_1 with

$$P(T_2 > t) = P(N_{T_1+t} - N_{T_1} = 0) = e^{-\lambda t}$$

so $T_2 \sim$ exponential(λ) with the remaining results for the T_i following similarly. ■

- put $S_n = T_1 + \cdots + T_n =$ *arrival time* for the n-th event $\sim$ gamma(n, λ)

Lemma VI.14 If $X \sim$ gamma(n, λ), then recalling $\Gamma(n) = (n-1)!$ for $x > 0$

$$P(X > x) = \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} e^{-\lambda x}.$$

Proof: We have

$$\begin{aligned} P(X > x) &= \int_x^\infty \frac{\lambda^n z^{n-1}}{(n-1)!} e^{-\lambda z}\, dz \\ &\text{using integration by parts with} \\ &u = z^{n-1}, du = (n-1)z^{n-2}, dv = e^{-\lambda z}, v = -e^{-\lambda z}/\lambda \\ = &\ \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} + \int_x^\infty \frac{\lambda^{n-1} z^{n-2}}{(n-2)!} e^{-\lambda z}\, dz \\ = &\ \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} + \frac{(\lambda x)^{n-2}}{(n-2)!} e^{-\lambda x} + \cdots + \frac{\lambda x}{1!} e^{-\lambda x} + \int_x^\infty \lambda e^{-\lambda z}\, dz \end{aligned}$$

which gives the result since $\int_x^\infty \lambda e^{-\lambda z}\, dz = e^{-\lambda x}$. ■

Proposition VI.15 The process $\{X_t : t \geq 0\}$ constructed from r.v.'s $T_1, T_2, \ldots \overset{i.i.d.}{\sim}$ exponential(λ) by (with $T_0 = 0$)

$$X_t = i \text{ when } S_i \leq t < S_{i+1}$$

is a Poisson process of intensity λ.

Proof: Using Lemma VI.14 $P(X_t \leq i) = P(S_{i+1} > t) = \sum_{k=0}^{i} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$ which is the cdf of the Poisson(λt). Further, if $t_1 < t_2$, then

$$P(X_{t_1} = i, X_{t_2} - X_{t_1} = j) = P(S_i \leq t_1 < S_{i+1}, S_{i+j} \leq t_2 < S_{i+j+1})$$

and if $j \neq 0, 1$, then

$$\begin{aligned} & P(S_i \leq t_1 < S_{i+1}, S_{i+j} \leq t_2 < S_{i+j+1}) \\ = \; & P\begin{pmatrix} U_1 = S_i \leq t_1, t_1 - U_1 < V_1 = T_{i+1} < t_2 - U_1, \\ U_2 = T_{i+2} + \cdots + T_{i+j} \leq t_2 - U_1 - V_1, \\ t_2 - U_1 - V_1 - U_2 < V_2 = T_{i+j+1} \end{pmatrix}. \end{aligned}$$

Now U_1, V_1, U_2, V_2 are mutually statistically independent with

$$\begin{aligned} U_1 &\sim \text{gamma}(i, \lambda), V_1 \sim \text{gamma}(1, \lambda) \\ U_2 &\sim \text{gamma}(j-1, \lambda), V_2 \sim \text{gamma}(1, \lambda). \end{aligned}$$

Therefore, with $g(\cdot, \alpha, \lambda)$ denoting the gamma(α, λ) density,

$$\begin{aligned} & P(X_{t_1} = i, X_{t_2} - X_{t_1} = j) \\ = & \int_0^{t_1} g_{i,\lambda}(u_2) \int_{t_1-u_1}^{t_2-u_1} g_{1,\lambda}(v_1) \int_0^{t_2-u_1-v_1} g_{j-1,\lambda}(u_2) \int_{t_2-u_1-v_1-u_2}^{\infty} g_{1,\lambda}(v_2) \\ & \qquad dv_2\, du_2\, dv_2\, du_1 \\ = & \frac{(\lambda t_1)^i}{i!} e^{-\lambda t_1} \frac{(\lambda(t_2 - t_1))^j}{j!} e^{-\lambda(t_2-t_1)} \text{ after doing the integration} \end{aligned}$$

and a similar result is obtained when $j = 0$ or $j = 1$. So $X_{t_1} \sim$ Poisson(λt_1) independent of $X_{t_2} - X_{t_1} \sim$ Poisson$(\lambda(t_2 - t_1))$ and this can be generalized to an arbitrary number of increments. ■

- this provides a way to simulate (approximately) a Poisson process of intensity λ

1. select n and generate $T_1, T_2, \ldots, T_n \overset{i.i.d.}{\sim}$ exponential(λ)
2. compute $S_1, S_2, \ldots, S_n$
3. compute $N_t = i$ when $S_i \leq t < S_{i+1}$

- note $E(S_n) = n/\lambda, Var(S_n) = n/\lambda^2$ will give you some idea of how big n has to be to cover say $[0, t]$ as

$$P(S_n > t) = P\left(\frac{\frac{1}{n}S_n - \frac{1}{\lambda}}{\sqrt{1/n\lambda^2}} > \frac{\frac{1}{n}t - \frac{1}{\lambda}}{\sqrt{1/n\lambda^2}}\right) = P\left(Z_n > \frac{\lambda t}{\sqrt{n}} - \sqrt{n}\right)$$

where $Z_n \xrightarrow{d} N(0,1)$ and $\lambda t/\sqrt{n} - \sqrt{n} \to -\infty$

Exercise VI.6 Simulate a Poisson process of intensity $\lambda = 1$ and plot the sample path on $[0, 50]$

Proposition VI.16 *(Superposition)* If $\{N_{i,t} : t \geq 0\}$ is a Poisson process of intensity λ_i for $i = 1, \ldots, k$, and these processes are mutually statistically independent, then $\{\sum_{i=1}^{k} N_{i,t} : t \geq 0\}$ is a Poisson process of intensity $\sum_{i=1}^{k} \lambda_i$.

Proof: **Exercise VI.7**

- note for $\delta >$ then $P(N_{t+\delta} - N_t = 0) = e^{-\lambda\delta} \to 1$ as $\delta \to 0$

Proposition VI.17 A Poisson process satisfies
(i) $P(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta)$
(ii) $P(N_{t+\delta} - N_t \geq 2) = o(\delta)$.

Proof:

$$\text{(i) } P(N_{t+\delta} - N_t = 1) = \lambda\delta e^{-\lambda\delta} = \lambda\delta \sum_{k=0}^{\infty} \frac{(-\lambda\delta)^k}{k!} = \lambda\delta + o(\delta),$$

$$
\begin{aligned}
\text{(ii) } & P(N_{t+\delta} - N_t \geq 2) = 1 - P(N_{t+\delta} - N_t = 0) - P(N_{t+\delta} - N_t = 1) \\
& = 1 - e^{-\lambda\delta} - \lambda\delta e^{-\lambda\delta} = 1 - \sum_{k=0}^{\infty} \frac{(-\lambda\delta)^k}{k!} - \lambda\delta \sum_{k=0}^{\infty} \frac{(-\lambda\delta)^k}{k!} \\
& = - \sum_{k=2}^{\infty} \left(\frac{(-\lambda\delta)^k}{k!} + \frac{(-\lambda\delta)^{k+1}}{(k+1)!} \right) = o(\delta).
\end{aligned}
$$

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- fact - any process satisfying Prop VI.17 and having independent increments is a Poisson process of intensity λ

- there are also inhomogeneous Poisson processes where the intensity depends on t

Example VI.5 (clumping)

- suppose $\{N_t : t \geq 0\}$ is a Poisson process of intenstity λ and $T = n\lambda$
- so in each subinterval $[0, \lambda), [\lambda, 2\lambda), \ldots, [(n-1)\lambda, n\lambda)$ we expect to see

$$E(N_{i\lambda} - N_{(i-1)\lambda}) = \lambda$$

events (one event when $\lambda = 1$)

- but for fixed $j > 0$

$$\begin{aligned} & P(N_{i\lambda} - N_{(i-1)\lambda} \geq j \text{ for some } i) \\ = \; & 1 - P(N_{i\lambda} - N_{(i-1)\lambda} < j \text{ for all } i) \\ \geq \; & 1 - \prod_{i=1}^{n} P(N_\lambda < j) \text{ where } N_\lambda \sim \text{ Poisson}(\lambda) \\ = \; & 1 - P^n(N_\lambda < j) \to 1 \text{ as } n \to \infty \end{aligned}$$

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Example VI.6 *(Waiting time paradox)*

- suppose buses arrive according to a fixed schedule as given by the i-th bus where buses arrive at times $1/\lambda, 2/\lambda, \ldots$

- you choose a time i/λ to catch the bus each day but arrive in the interval $[(i-1)/\lambda, i/\lambda]$ according to a uniform distribution in the interval so on average you will wait a period $i/\lambda - ((i-1)/\lambda + i/\lambda)/2 = 1/2\lambda$ for the next bus

- suppose there are n buses during the day and clearly the average time between buses is $1/\lambda$

- but now suppose (unrealistically) buses arrive according to arrival times $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Uniform}(0, n/\lambda)$

- let $N_{t,n} = \#\{X_i : X_i \le t\} = \#$ of arrivals before time $t \sim$ binomial(n, p_n) for $0 \le t < n/\lambda$ and $p_n = \lambda t/n$ so the expected number of buses arriving in the interval $[0, t]$ is λt (and so the expected number of buses arriving in the interval $[(i-1)/\lambda, i/\lambda]$ is 1

- note that the conditions for convergence of the binomial to the Poisson apply so $N_{t,n} \xrightarrow{d} N_t \sim \text{Poisson}(\lambda t)$ and for large n we can consider buses arriving (approximately) according to a Poisson process of intensity λ

- suppose you begin waiting at time t and concern is with how long you need to wait on average until the next bus arrives

- then $\{N_{t+s} - N_t : s \geq 0\}$ is an approximate Poisson process of intensity λ, which means that the interarrival times for this process are $i.i.d.$ exponential(λ), and so the mean waiting time for the next bus after time t is $1/\lambda$, which is twice as long as when the buses are on a fixed schedule!

- note that this reflects the *memoryless feature* of the exponential distribution for if $T \sim \text{exponential}(\lambda)$, then for $a, b \geq 0$

$$P(T > b + a \,|\, T > a) = \frac{P(T > b + a\,,\, T > a)}{P(T > a)}$$

$$= \frac{P(T > b + a\,)}{P(T > a)} = \frac{e^{-\lambda(b+a)}}{e^{-\lambda a}} = e^{-\lambda b}$$

so the probability you will wait an additional b time units doesn't depend on the fact you have already waited a time units

Proposition VI.18 *(Thinning)* Suppose $\{N_t : t \geq 0\}$ is a Poisson process of intensity λ and each arrival is labeled i with probability p_i where $\sum p_i = 1$ and let $\{N_{i,t} : t \geq 0\}$ denote the process counting the number of arrivals labeled i. Then $\{N_{i,t} : t \geq 0\}$ is a Poisson process of intensity $p_i\lambda$ and these processes are mutually statistically independent.

Proof: Consider interarrival times $T_1, T_2, \ldots \overset{i.i.d.}{\sim}$ exponential(λ) and labels $L_1, L_2, \ldots \overset{i.i.d.}{\sim} (p_1, p_2, \ldots)$ so $(L_1, T_1), (L_2, T_2), \ldots$ are $i.i.d.$ Since the process $\{N_{i,t} : t \geq 0\}$ is constructed from those pairs with label i this implies that the $\{N_{i,t} : t \geq 0\}$ processes are mutually statistically independent. Also increments from the individual processes are constructed from separate groups of (labeled) arrival times and so are independent. Now if $s < t$, then for the process labelled 1,

$$P(N_{1,t}=j)=\sum_{k=j}^{\infty}P(N_{1,t}=j\,|\,N_t=k)P(N_t=k)$$

$$\begin{aligned}
&= \sum_{k=j}^{\infty}\binom{k}{j}p_1^j(1-p_1)^{k-j}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\\
&= \frac{(\lambda t p_1)^j}{j!}e^{-\lambda t}\sum_{k=j}^{\infty}\frac{(\lambda(1-p_1)t)^{k-j}}{(k-j)!}\\
&= \frac{(\lambda t p_1)^j}{j!}e^{-\lambda t}e^{\lambda(1-p_i)t}=\frac{(p_1\lambda t)^j}{j!}e^{-p_1\lambda t}
\end{aligned}$$

and

$$P(N_{1,t+s}-N_{1,t}=j)=\sum_{k=0}^{\infty}P(N_{1,t+s}-N_{1,t}=j,N_{1,t}=k)$$

$$= \sum_{k=0}^{\infty}\sum_{n=j}^{\infty}\sum_{m=k}^{\infty}\left\{\begin{array}{l} P(N_{1,t+s}=k+j,N_{1,t}=k\,|\,N_t=m,N_{t+s}=m+n) \\ \times P(N_t=m,N_{t+s}-N_t=n)\end{array}\right\}$$

$$= \sum_{k=0}^{\infty}\sum_{n=j}^{\infty}\sum_{m=k}^{\infty}\begin{array}{l}\left\{\binom{m}{k}p_1^k(1-p_1)^{m-k}\frac{(\lambda t)^m}{m!}e^{-\lambda t}\right\} \\ \times\left\{\binom{n}{j}p_1^j(1-p_1)^{n-k}e^{-\lambda t}\frac{(\lambda s)^n}{n!}e^{-\lambda s}\right\}\end{array}$$

$$= \sum_{k=0}^{\infty}\frac{(p_1\lambda t)^k}{k!}e^{-p_1\lambda t}\frac{(p_1\lambda s)^j}{j!}e^{-p_1\lambda s}=\frac{(p_1\lambda s)^j}{j!}e^{-p_1\lambda s}$$

so the increments are Poisson of intensity $p_1\lambda$ and the same argument applies to any of the labelled processes. ■

- a Poisson process is a particular example of what is known as a *counting process* as it counts the number of events occurring randomly in any subset (a, b) of $[0, \infty)$

- you can also have processes that count the number of events occurring randomly in any subset of a general set S (e.g. R^1, R^2, R^3, S^1, S^2, etc.)

- if there is a measure μ on S (e.g. volume measure) we say the process is a Poisson process of intensity λ if, whenever $A_1, A_2, \ldots \subset S$ satisfy $\mu(A_i) < \infty$ and $A_i \cap A_j = \phi$ for all i and $j \neq i$, then the counts are mutually staistically independent and such that the count for A_i is distributed Poisson$(\lambda\mu(A_i))$

Exercise VI.8 Text 4.3.7

Exercise VI.9 Text 4.3.8

Exercise VI.10 Text 4.3.18