Probability and Stochastic Processes I I - Lecture 6a

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VI Continuous Processes

- we consider processes where the time domain T and/or the state space is a continuous set like $T = [0, \infty)$ or $T = R^1$

VI.1 Brownian Motion (Wiener Process)

Definition VI.1 A s.p. $\{W_t : t \geq 0\}$ is a *Brownian motion* (*standard Wiener process*) if (i) $P(W_0 = 0) = 1$ (ii) the process has *independent increments*, namely, for any $0 < t_1 < \cdots < t_k$ then $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}$ are mutually stat. ind. and (iii) $W_t - W_s \sim N(0, t-s)$ for any $0 \leq s \leq t$. ■

- then $\{X_t : t \geq 0\}$ with $X_t = \tau W_t \sim N(0, \tau^2(t-s))$ is a general Brownian motion

Proposition VI.1 $\{X_t : t \geq 0\}$ is a Gaussian process with mean function 0 and autocovariance function $\sigma(s,t) = \tau^2 \min(s,t)$.

Proof: For any $0 = t_0 < t_1 < \cdots < t_n$ and $c_1, \ldots, c_n \in R^1$

$$\begin{aligned} & \sum_{i=1}^{n} c_i X_{t_i} = \tau \sum_{i=1}^{n} c_i W_{t_i} = \tau [c_n (W_{t_n} - W_{t_{n-1}}) + \\ & (c_{n-1} + c_n)(W_{t_{n-1}} - W_{t_{n-2}}) + \cdots + (c_1 + \cdots + c_n) W_{t_1} \\ \sim \;\; & N\left(0, \tau^2 \sum_{i=1}^{n} \left(\sum_{j=1}^{n-i+1} c_{n-j+1} \right)^2 (t_i - t_{i-1}) \right) \end{aligned}$$

and so $(X_{t_1}, \ldots, X_{t_n})'$ is multivariate normal since every linear combination is normal (Prop. III.9.8 in STAC62). Also,

$$\begin{aligned} & \sigma(s,t) = E(X_s X_t) = \tau^2 E(W_s W_t) \overset{s \leq t}{=} \tau^2 E(W_s (W_s + W_t - W_s)) \\ = \;\; & \tau^2 E(W_s^2) + \tau^2 E(W_s (W_t - W_s)) = \tau^2 s + \tau^2 0 = \tau^2 s = \tau^2 \min(s,t). \end{aligned}$$

Therefore, $(X_{t_1}, \ldots, X_{t_n})' \sim N_n(\mathbf{0}, \tau^2 (\min(t_i, t_j)))$ and so by KCT this is a Gaussian process. ■

Proposition VI.2 There exists a version of $\{W_t : t \geq 0\}$ also satisfying (iv) $P(W_t$ is continuous in $t) = 1$ and (v) $P(W_t$ is nowhere differentiable in $t) = 1$.
Proof: Accept.

- we restrict attention hereafter to a version where Prop. VI.2 applies

Example VI.1

- let $\{W_t : t \geq 0\}$ be a Brownian motion and define $\{Y_t : t \geq 0\}$ by $Y_t = \alpha W_{t/\alpha^2}$ then

(i) $Y_0 = 0$
(ii) or any $0 < t_1 < \cdots < t_k$ then $Y_{t_1} = \alpha W_{t_1/\alpha^2}$, $Y_{t_2} - Y_{t_1} = \alpha(W_{t_2/\alpha^2} - W_{t_{21}/\alpha^2}), \ldots$ are mutually stat. ind. and
(iii) $Y_t - Y_s \sim N(0, \alpha^2(t/\alpha^2 - s/\alpha^2)) = N(0, t-s)$ for any $0 \leq s \leq t$
(iv) the sample paths of $\{Y_t : t \geq 0\}$ are continuous

- in other words a Brownian motion is defined in terms of its distributional properties and there are many stochastic processes that satisfy these
■

- how does Brownian motion arise? as a limiting process

- suppose $Z_1, Z_2, \ldots \overset{i.i.d.}{\sim} -1 + 2\text{Bernoulli}(1/2)$ with $E(Z_i) = 0$, $Var(Z_i) = 1$ and put $S_0 = 0$, $S_n = \sum_{i=1}^{n} Z_i$ a ssrw

Proposition VI.3 (*Donsker's Theorem or Invariance Principle*) As $n \to \infty$,

$$\left\{ n^{-1/2} S_{\lfloor nt \rfloor} : t \in [0,1] \right\} \xrightarrow{d} \{ W_t : t \in [0,1] \}$$

- space is shrunk by factor $1/\sqrt{n}$ and time speeded up by factor n

- note when $t = k/n$ for $k \in \{0, \ldots, n-1\}$, then

$$\begin{aligned} n^{-1/2} S_{\lfloor nt \rfloor} &= n^{-1/2} S_k = n^{-1/2} \sum_{i=1}^{k} Z_i = \left(\frac{k}{n} \right)^{1/2} \sqrt{k} \left(\frac{1}{k} \sum_{i=1}^{k} Z_i \right) \\ &= t^{1/2} \sqrt{k} \left(\frac{1}{k} \sum_{i=1}^{k} Z_i \right) \xrightarrow{d} t^{1/2} N(0,1) = N(0,t) \\ \text{as } n, k &\to \infty \text{ with } t = k/n \text{ fixed (by the CLT)} \end{aligned}$$

- sample paths $t \to n^{-1/2} S_{\lfloor nt \rfloor}$ are not continuous but for $k \in \{0, \ldots, n-1\}$

$$t \to n^{-1/2} \left[(1 - nt + k) S_{\lfloor nt \rfloor} + (nt - k) S_{\lfloor nt \rfloor + 1} \right] \text{ for } t \in \left[\frac{k}{n}, \frac{k+1}{n} \right]$$

has continuous sample paths and the same convergence result applies and note no derivative exists at each endpoint $t = k/n$ or $t = (k+1)/n$

- for a Brownian motion on $[0, t_0]$ then

$$\left\{ (t_0/n)^{1/2} S_{\lfloor nt/t_0 \rfloor} : t \in [0, t_0] \right\} \xrightarrow{d} \{ (W_t : t \in [0, t_0] \}$$

and for $k \in \{0, \ldots, n-1\}$

$$t \to (t_0/n)^{1/2} \left[(1 - nt/t_0 + k) S_{\lfloor nt/t_0 \rfloor} + (nt/t_0 - k) S_{\lfloor nt/t_0 \rfloor + 1} \right] \text{ for } t \in \left[\frac{k}{n} t_0, \frac{k+1}{n} t_0 \right]$$

has continuous sample paths

- these results also tell us how to simulate (approximately) from $\{W_t : t \in [0, t_0]\}$

(1) choose n, generate $U_1, \ldots, U_n \overset{i.i.d.}{\sim}$ Bernoulli$(1/2)$ with $Z_i = -1 + 2U_i$,

(2) for $k \in \{1, \ldots, n\}$ compute $S_k = \sum_{i=1}^{k} Z_i$ and

(3) $W_t = (t_0/n)^{1/2} [(1 - nt/t_0 + k)S_k + (nt/t_0 - k)S_{k+1}]$ for $t \in \left[\frac{k}{n}t_0, \frac{k+1}{n}t_0\right]$ for $k \in \{0, \ldots, n-1\}$

Exercise VI.1 Generate two (approximate) sample paths for a Brownian motion on $[0, 2.5]$ and plot them.

Definition VI.2 A process $\{X_t : t \in T\}$ with $T \subset R^1$ is a *Markov process* if it satisfies $X_{t_n} \,|\, X_{t_1}, \ldots, X_{t_{n-1}} \sim X_{t_n} \,|\, X_{t_{n-1}}$ for all n and times $t_1 < \cdots < t_n$.

Proposition VI.5 A general Bm $\{X_t : t \geq 0\}$ is a Markov process.

Proof: We have that

$$\begin{aligned} & P(X_{t_n} \leq x_n \,|\, X_{t_1} = x_1, \ldots, X_{t_{n-1}} = x_{n-1}) \\ = \;& P(X_{t_n} - X_{t_{n-1}} \leq x_n - x_{n-1} \,|\, X_{t_1} - X_0 = x_1, \ldots, \\ & \qquad X_{t_{n-1}} - X_{t_{n-2}} = x_{n-1} - x_{n-2}) \\ = \;& P(X_{t_n} - X_{t_{n-1}} \leq x_n - x_{n-1}) \text{ by independent increments} \\ = \;& \Phi(\tau^{-1/2}(t_n - t_{n-1})^{-1/2}(x_n - x_{n-1})) \\ = \;& P(X_{t_n} \leq x_n \,|\, X_{t_{n-1}} = x_{n-1}) \text{ by independent increments} \end{aligned}$$

using increments $X_{t_{n-1}} - X_0, X_{t_n} - X_{t_{n-1}}$. ■

Definition VI.3 A process $\{X_t : t \in T\}$ with $T \subset R^1$ is a *martingale* if $E(|X_t|) < \infty$ for all t and

$$E(X_{t_n} \,|\, X_{t_1}, \ldots, X_{t_{n-1}}) = X_{t_{n-1}}$$

for all n and times $t_1 < \cdots < t_n$.

Proposition VI.5 A general Bm $\{X_t : t \geq 0\}$ is a martingale.

Proof: We have that $X_{t_n} - X_{t_{n-1}}$ is statistically independent of the random vector $(X_{t_1}, \ldots, X_{t_{n-1}})$ for every n and $t_1 < \cdots < t_n$ and so

$$\begin{aligned} & E(X_t \,|\, X_{t_1}, \ldots, X_{t_{n-1}}) = E(X_{t_{n-1}} + (X_t - X_{t_{n-1}}) \,|\, X_{t_1}, \ldots, X_{t_{n-1}}) \\ = \;& X_{t_{n-1}} + E(X_t - X_{t_{n-1}}) = X_{t_{n-1}}. \end{aligned}$$

■

- note that the definitions of Markov process and martingale also apply to conditioning on uncountably many values such as $\{X_s : \text{for } 0 \leq s \leq s_0\}$ where $s_0 < t$

- $T : [0, \infty)$ is a stopping time for the process $\{X_t : t \geq 0\}$ if

$$\{T \leq t\} \in \mathcal{A}_{\{X_s : 0 \leq s \leq t\}}$$

- $T_b = \inf\{s : X_s = b\}$ is a stopping time, called the *first passage time* of b, since for $b > 0$

$$\{T_b \leq t\} = \{T_b > t\}^c = \{X_s < b \text{ for } s \leq t\}^c \in \mathcal{A}_{\{X_s : 0 \leq s \leq t\}}$$

and this makes sense because X_t has continuous sample paths so if ever

$X_s(\omega) \geq b$ the sample path must have hit b and similarly when $b < 0$

- note let $q_n \in \mathbb{Q}$ be s.t. $q_n \to t$ so by the cty of sample paths $X_{q_n}(\omega) \to X_t(\omega)$ so the sample paths are determined at the rational times which are countable

- we say $\{X_t : t \geq 0\}$ has the *strong Markov property* whenever $X_{T+t} \,|\, \{X_s : s \leq T\} \sim X_{T+t} \,|\, X_T$ for any stopping time T for the process

- fact: it can be proven that a Bm has the strong Markov property and that $X_{T+t} - X_T \,|\, X_T = b$ is a Bm independent of $\{X_s : 0 \leq s \leq T\}$

Proposition VI.6 For Bm $\{W_t : t \geq 0\}$ then $P(T_b < \infty) = 1$.
Proof: Since the sample paths of $\{W_t : t \geq 0\}$ are continuous, for $b > 0$

$$
\begin{aligned}
p = P(T_b < \infty) &= \lim_{t\to\infty} P(T_b < t) \geq \lim_{t\to\infty} P(W_t > b) \\
&= \lim_{t\to\infty} P(\sqrt{t}Z > b) \text{ where } Z \sim N(0,1) \\
&= \lim_{t\to\infty} (1 - \Phi(b/\sqrt{t})) = 1/2
\end{aligned}
$$

and so this inequality holds for any b.

Then by the Strong Markov property for $t' < t$

$$\begin{aligned}
& P(T_{2b} < t \mid T_b = t') \\
= \;& P(W_{T_b+s} = 2b \text{ for some } s \in (t', t) \mid W_s < b \text{ for } s < t' = T_b) \\
= \;& P(W_{T_b+s} - W_{T_b} = b \text{ for some } s \in (t', t) \mid W_s < b \text{ for } s < t' = T_b) \\
= \;& P(T_b < t - t').
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(T_{2b} < \infty) &= \lim_{t\to\infty} P(T_{2b} < t) \\
&= \lim_{t\to\infty} \int_0^t P(T_{2b} < t \mid T_b = t')\, P_{T_b}(dt') \\
&= \lim_{t\to\infty} \int_0^t P(T_b < t - t')\, P_{T_b}(dt') \\
&= \lim_{t\to\infty} \int_0^\infty P(T_b < t - t') I_{(0,t]}(t')\, P_{T_b}(dt') \text{ and by MCT} \\
&= \int_0^\infty \lim_{t\to\infty} P(T_b < t - t') I_{(0,t]}(t')\, P_{T_b}(dt') = p^2 \geq 1/2
\end{aligned}$$

and by repeating this argument $p^n = \lim_{t\to\infty} P(T_{nb} < t) \geq 1/2$. This implies $p = 1$ otherwise $p^n \to 0$ as $n \to \infty$. ■

Lemma VI.7 If $X \sim N(0, \sigma^2)$, then $Y = |X|$ has density $(2/\sigma)\varphi(y/\sigma)$ for $y \geq 0$.

Proof: For $y \geq 0$ the cdf of Y is given by

$$P(Y \leq y) = P(-y \leq X \leq y) = \Phi(y/\sigma) - \Phi(-y/\sigma) = 2\Phi(y/\sigma) - 1$$

and so the pdf of Y is $dP(Y \leq y)/dy = (2/\sigma)\varphi(y/\sigma)$. ■

- note -

$$P(Y \geq y) = 1 - (2\Phi(y/\sigma) - 1) = 2(1 - \Phi(y/\sigma)) = 2P(X \geq y)$$

Proposition VI.8 For Bm $\{W_t : t \geq 0\}$ then $M_t = \max\{W_s : 0 \leq s \leq t\}$ has the same distribution as $|W_t|$ with density, for $m \geq 0$,

$$f_{M_t}(m) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{m^2}{2t}\right).$$

Proof: Suppose $m > 0$, then

$$P(M_t \geq m) = P(M_t \geq m, W_t - m \geq 0) + P(M_t \geq m, W_t - m < 0).$$

and using $\{T_m \leq t\} = \{M_t \geq m\}$ we have

$$P(M_t \geq m, W_t - m < 0) = P(W_t - W_{T_m} < 0, T_m \leq t).$$

Then since the distribution of $W_t - W_{T_m} \,|\, T_m = t'$, when $t > t'$, is symmetrical about 0

$$\begin{aligned} & P(M_t \geq m, W_t - m < 0) = P(W_t - W_{T_m} < 0 \,|\, T_m \leq t) P(T_m \leq t) \\ = \; & P(W_t - W_{T_m} \geq 0 \,|\, T_m \leq t) P(T_m \leq t) = P(M_t \geq m, W_t - m \geq 0). \end{aligned}$$

Therefore

$$P(M_t \geq m) = 2P(M_t \geq m, W_t \geq m) = 2P(W_t \geq m)$$

since $\{W_t \geq m\} \subset \{M_t \geq m\}$. This gives the result since $W_t \sim N(0, t)$.

Proposition VI.9 For Brownian motion $\{W_t : t \geq 0\}$ then T_b has density

$$f_{T_b}(t) = \frac{|b|}{\sqrt{2\pi t^3}} \exp\left(-\frac{b^2}{2t}\right).$$

- this is an example of the *inverse Gaussian (or Wald) distribution* which in general has density on $(0, \infty)$

$$f_{\mu,\lambda}(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda}{2t}\left(\frac{t-\mu}{\mu}\right)^2\right)$$

with $\mu, \lambda > 0$ denoted $IG(\mu, \lambda)$, so $T_b \sim IG(\infty, b^2)$.

- a plot with $b = 2$ and note the long tail

- in fact

$$\begin{aligned} E(T_b) &= \int_0^\infty t \frac{|b|}{\sqrt{2\pi t^3}} \exp\left(-\frac{b^2}{2t}\right) dt \text{ putting } u = 1/t \\ &= \frac{|b|}{\sqrt{2\pi}} \int_0^\infty u^{-3/2} \exp\left(-\frac{b^2}{2}u\right) du = \infty \end{aligned}$$

Proof of Prop. VI.9: We have from Prop. VI.8

$$P(T_b \le t) = P(M_t \ge b) = \int_b^\infty \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{m^2}{2t}\right) dm$$

and making the change of variable $m \to u = b^2 t/m^2$

$$= \int_0^t \frac{|b|}{\sqrt{2\pi u^3}} \exp\left(-\frac{b^2}{2u}\right) du$$

which implies the result by differentiating wrt to t. ■

Example VI.2

- for $a, b > 0$ put $T_{-a,b} = \min(T_{-a}, T_b)$ which is a stopping time and

$$\begin{aligned} P(T_{-a,b} &< \infty) = P(T_{-a} < \infty \text{ or } T_b < \infty) = 1 - P(T_{-a} = \infty, T_b = \infty) \\ &\geq 1 - P(T_{-a} = \infty) = 1. \end{aligned}$$

- since Bm is a martingale and $|W_t| I_{\{T_{-a,b} \geq t\}} \leq \max(a, b)$ so Prop. V.10 (Optional Stopping Corollary) applies (suitably generalized) which implies

$$\begin{aligned} E(W_{T_{-a,b}}) &= E(W_0) = 0 = P(W_{T_{-a,b}} = -a)(-a) + P(W_{T_{-a,b}} = b)b \\ &= (1 - P(W_{T_{-a,b}} = b))(-a) + P(W_{T_{-a,b}} = b)b \end{aligned}$$

which implies

$$P(W_{T_{-a,b}} = b) = \frac{a}{a + b}$$

■

Proposition VI.10 If $Y_t = W_t^2 - t$, then $\{Y_t : t \geq 0\}$ is a martingale.

Proof: For $0 < t < s$ and using the Strong Markov property with stopping time $T \equiv t$ and Prop. VI.5

$$\begin{aligned}
& E(Y_s \,|\, \{W_r : r \leq t\}) = E(W_s^2 - s \,|\, \{W_r : r \leq t\}) \\
= \;& Var(W_s \,|\, \{W_r : r \leq t\}) + (E(W_s \,|\, \{W_r : r \leq t\}))^2 - s \\
= \;& Var((W_s - W_t) + W_t \,|\, \{W_r : r \leq t\}) + (E(W_s \,|\, \{W_r : r \leq t\}))^2 - s \\
= \;& (s - t) + W_t^2 - s = W_t^2 - t = Y_t.
\end{aligned}$$

Now $A_{\{Y_r : r \leq t\}} \subset A_{\{W_r : r \leq t\}}$ and therefore by Prop. V.14

$$\begin{aligned}
& E(Y_s \,|\, \{Y_r : r \leq t\}) = E(E(Y_s \,|\, \{W_r : r \leq t\}) \,|\, \{Y_r : r \leq t\}\} \\
= \;& E(E(Y_t \,|\, \{Y_r : r \leq t\}\} = Y_t
\end{aligned}$$

■

Example VI.3

- for $a, b > 0$ put $T_{-a,b} = \min(T_{-a}, T_b)$ which is a stopping time for$\{W_t : t \geq 0\}$

- then

$$E(Y_{T_{-a,b}}) = E(W^2_{T_{-a,b}} - T_{-a,b}) = E(W^2_{T_{-a,b}}) - E(T_{-a,b})$$
$$= \frac{a^2 b}{a+b} + \frac{ab^2}{a+b} - E(T_{-a,b}) = ab - E(T_{-a,b})$$

which establishes

$$E(T_{-a,b}) = ab$$

provided $E(Y_{T_{-a,b}}) = E(Y_0) = 0$

- now $T_m = \min(T_{-a,b}, m)$ is a bounded stopping time and $\{Y_t : t \geq 0\}$ is a martingale and so $E(Y_{T_m}) = E(Y_0) = 0$ by the Optional Stopping lemma, Prop V.8 and so

$$E(Y_{T_m}) = E(W^2_{T_m}) - E(T_m) = 0$$

- then $T_m \overset{wp1}{\rightarrow} T_{-a,b}$ as $m \longrightarrow \infty$ which implies

$$\begin{aligned} E(T) &= \lim_{m\to\infty} E(T_m) \text{ by MCT} \\ &= \lim_{m\to\infty} E(W_{T_m}^2) \\ &= E(W_{T_{-a,b}}^2) = ab \text{ by DCT} \end{aligned}$$

since $W_{T_m}^2 \overset{wp1}{\rightarrow} W_{T_{-a,b}}^2$ and $W_{T_{-a,b}}^2 \leq \max(a^2, b^2)$ and $E(W_{T_{-a,b}}^2) = ab$
■

Definition VI.4 A process $\{X_t : t \geq 0\}$ s.t. $X_t = x_0 + \mu t + \sigma W_t$ is called a *diffusion* with *initial value* x_0, *drift* μ and *volatility* σ.

- sometimes this is denoted by $X_0 = x_0$ and $dX_t = \mu dt + \sigma dW_t$

Proposition VI.11 If $\{X_t : t \geq 0\}$ is a diffusion then $E(X_t) = x_0 + \mu t$, $Var(X_t) = \sigma^2 t$ and $Cov(X_s, X_t) = \sigma^2 \min(s, t)$ and

$$X_t - X_s \sim N(\mu(t - s), \sigma^2(t - s))$$

and nonoverlapping increments are mutually statistically independent.

Exercise VI.2 Text 4.1.4

Exercise VI.3 Text 4.1.6

Exercise VI.4 Text 4.1.11

Exercise VI.5 Text 4.1.12

Example VI.4 *Stock options - continuous*

$$X_t = \text{price of stock at time } t \geq 0 = x_0 \exp(\mu t + \sigma W_t)$$

- so $Y_t = \log x_0 + \mu t + \sigma W_t$ is a diffusion

- this is an assumption here but it can be justified by looking at the limit of a model (the binomial model) where there are n time steps and the stock goes up by a fixed factor $u > 1$ or down by a fixed factor $d < 1$ at each time step, so the stock price is ux_0 or dx_0 where $u = 1 + r_u, d = 1 - r_d$

- relating this to the discrete case the martingale probability that the stock goes up in one period is

$$\frac{x_0 - (1 - r_d)x_0}{(1 + r_u)x_0 - (1 - r_d)x_0} = \frac{r_d}{r_u + r_d}$$

- but a more complete analysis also takes into account the *risk-free rate* r where no arbitrage implies $r_d < r < r_u$ and in that case the martingale probability that the stock goes up is $(r - r_d)/(r_u + r_d)$

- after one year 1 unit grows to $(1+r)$, if instead it compounds n times during the year then 1 unit grows to $(1+r/n)^n \to e^r$ and if this is done for t years 1 unit grows to $(1+r/n)^{nt} \to e^{rt}$ and $t > 0$

- since $1/n \to 0$ this referred to as *continuous compounding*

- so a stock price X_t at future time t has *present value*

$$D_t = e^{-rt}X_t = e^{-rt}x_0 \exp(\mu t + \sigma W_t) = x_0 \exp((\mu - r)t + \sigma W_t)$$

- what price C should be paid for a (European) call with strike price K at strike time S?

- when is the process $\{D_t : t \geq 0\}$ a martingale, for $s_0 < t$?

$$\begin{aligned}
& E(D_t \mid \{D_s : 0 \leq s \leq s_0\}) = D_{s_0} E(D_t / D_{s_0} \mid \{D_s : 0 \leq s \leq s_0\}\} \\
= \;& D_{s_0} E(\exp((\mu - r)(t - s_0) + \sigma(W_t - W_{s_0}) \mid \{W_s : 0 \leq s \leq s_0\}) \\
= \;& D_{s_0} \exp((\mu - r)(t - s_0)) E(\exp(\sigma(W_t - W_{s_0})) \\
& \text{and } E(\exp(\sigma(W_t - W_{s_0})) = \text{ mgf of } N(0, t - s_0) \text{ r.v. evalated at } \sigma \\
= \;& D_{s_0} \exp((\mu - r)(t - s_0)) \exp(\sigma^2(t - s_0)/2) = D_{s_0} \exp((\mu - r + \sigma^2/2) \\
= \;& D_{s_0} \text{ iff } \mu = r - \sigma^2/2.
\end{aligned}$$

Proposition VI.11 *(Black-Scholes)* Under the martingale probability distribution the no-arbitrage price for the call is given by

$$\begin{aligned} C &= E(e^{-rS}\max(0, X_S - K)) = x_0\Phi\left(\frac{(r+\sigma^2/2)S - \log(K/x_0)}{\sigma S^{1/2}}\right) - \\ &\qquad e^{-rS}K\Phi\left(\frac{(r-\sigma^2/2)S - \log(K/x_0)}{\sigma S^{1/2}}\right) \end{aligned}$$

Proof: We have $E(e^{-rS}\max(0, X_S - K)) = e^{-rS}E(\max(0, X_S - K))$ and

$$\begin{aligned} & E(\max(0, X_S - K)) = \int_K^\infty (x - K)\, P_{X_S}(dx) \\ = & \int_K^\infty x\, P_{X_S}(dx) - KP_{X_S}(X_S > K) \\ & P_{X_S}(X_S > K) = P(x_0 \exp(\mu S + \sigma W_S) > K) \\ = & P\left(W_S > \frac{\log(K/x_0) - \mu S}{\sigma}\right) = P\left(W_S < \frac{\mu S - \log(K/x_0)}{\sigma}\right) \\ = & \Phi\left(\frac{\mu S - \log(K/x_0)}{\sigma S^{1/2}}\right) \text{ since } W_S \sim N(0, S). \end{aligned}$$

Since $X_S > K$ iff $x_0 \exp(\mu S + \sigma W_S) > K$ iff $W_S > (\log(K/x_0) - \mu S)/\sigma$

$$
\begin{aligned}
& \int_{(\log(K/x_0)-\mu S)/\sigma}^{\infty} x_0 \exp(\mu S + \sigma w)\, P_{W_S}(dw) \\
= \;& x_0 \exp(\mu S) \int_{(\log(K/x_0)-\mu S)/\sigma}^{\infty} \exp(\sigma w)\, P_{W_S}(dw) \\
& \int_{(\log(K/x_0)-\mu S)/\sigma}^{\infty} \exp(\sigma w)\, P_{W_S}(dw) \\
= \;& \int_{(\log(K/x_0)-\mu S)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi S}} \exp\left(\sigma w - \frac{w^2}{2S}\right) dw \\
= \;& \exp(\sigma^2 S/2) \int_{(\log(K/x_0)-\mu S)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi S}} \exp\left(-\frac{(w-\sigma S)^2}{2S}\right) dw \\
= \;& \exp(\sigma^2 S/2) \left\{1 - \Phi\left(\frac{\log(K/x_0) - \mu S - \sigma^2 S}{\sigma S^{1/2}}\right)\right\} \\
= \;& \exp(\sigma^2 S/2)\Phi\left(\frac{(\mu + \sigma^2)S - \log(K/x_0)}{\sigma S^{1/2}}\right)
\end{aligned}
$$

and therefore

$$\begin{aligned} &\int_{(\log(K/x_0)-\mu S)/\sigma}^{\infty} x_0 \exp(\mu S + \sigma w)\, P_{W_S}(dw) \\ = \quad & x_0 \exp(\mu S + \sigma^2 S/2)\Phi\left(\frac{(\mu+\sigma^2)S - \log(K/x_0)}{\sigma S^{1/2}}\right) \\ = \quad & x_0 \exp(rS)\Phi\left(\frac{(r+\sigma^2/2)S - \log(K/x_0)}{\sigma S^{1/2}}\right) \end{aligned}$$

which completes the proof. ■