Probability and Stochastic Processes II - Lecture 5c

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- the following result is concerned with stopping times for random walks

Proposition V.11 *(Wald's Theorem)* Suppose $X_n = \sum_{i=0}^{n} Z_i$ where $Z_0 = a$ and $Z_1, Z_2, \ldots$ are $i.i.d.$ with finite mean m. Let T be a stopping time for $\{X_n : n \in \mathbb{N}_0\}$ with $E(T) < \infty$. Then $E(X_T) = a + mE(T)$.

Proof: Note that $P(T < \infty) = 1$, otherwise $E(T) = \infty$. Also, $\{T = t\} = (X_0, \ldots, X_t)^{-1} B_t$ for some $B_t \in \mathcal{B}^{t+1}$ and so $\{T = t\} = (Z_0, \ldots, Z_t)^{-1} B_t'$ for some $B_t' \in \mathcal{B}^{t+1}$.

Now supposing $E(X_T - a)$ exists then

$$E(X_T - a) = E\left(\sum_{i=1}^{T} Z_i\right) = E\left(\sum_{i=1}^{\infty} Z_i I_{T \geq i}\right) = E\left(\lim_{n\to\infty} \sum_{i=1}^{n} Z_i I_{T \geq i}\right).$$

We have

$$\left|\sum_{i=1}^{n} Z_i I_{T \geq i}\right| \leq \sum_{i=1}^{n} |Z_i|\, I_{T \geq i} \leq \sum_{i=1}^{\infty} |Z_i|\, I_{T \geq i}$$

and so

$$E\left(\sum_{i=1}^{\infty}|Z_i|\,I_{T\geq i}\right)=\sum_{i=1}^{\infty}E\left(|Z_i|\,I_{T\geq i}\right)=\sum_{i=1}^{\infty}E(|Z_i|)P(T\geq i)$$

$$=\quad E(|Z_1|)\sum_{i=1}^{\infty}P(T\geq i)=E(|Z_1|)E(T)<\infty$$

since $\{T\geq i\}=\{T\leq i-1\}^c\in\mathcal{A}_{Z_0,\ldots,Z_{i-1}}$ and $|Z_i|$ and $I_{T\geq i}$ are statistically independent so

$$E\left(|Z_i|\,I_{T\geq i}\right)=E\left(|Z_i|\right)E(I_{T\geq i})=E\left(|Z_i|\right)P(T\geq i).$$

Therefore by DCT

$$E(X_T-a)=E\left(\lim_{n\to\infty}\sum_{i=1}^{n}Z_iI_{T\geq i}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}E\left(Z_iI_{T\geq i}\right)$$

$$=\quad\sum_{i=1}^{\infty}E\left(Z_iI_{T\geq i}\right)=\sum_{i=1}^{\infty}E\left(Z_i\right)E(I_{T\geq i})=mE(T).$$

This implies that $E(X_T-a)$ exist finitely which implies $E(X_T)$ is finite and so $E(X_T)=a+mE(T)$ as required. ■

Example V.8 *Random walks*

- $X_n = \sum_{i=0}^{n} Z_i$ where $Z_0 = a$ and $Z_1, Z_2, \ldots$ are $i.i.d.$ with mean 0, then X_n is a martingale

- Wald's theorem establishes that $E(X_T) = E(X_0)$ and these are different conditions than the various optional stopping results that were proved required ■

Example V.9

- let $Z_1, Z_2, \ldots$ be the outcomes from independent rolls of a fair die so $Z_i \in \{1, 2, 3, 4, 5, 6\}$ and $E(Z_i) = 3.5$

- let $X_n = \sum_{i=1}^{n} Z_i$

- let $R = \inf\{n \geq 1 : Z_n = 5\}$ so

$$\begin{aligned} E(R) &= \sum_{i=1}^{\infty} P(R \geq i) = \sum_{i=1}^{\infty} P(R \geq i) = \sum_{i=1}^{\infty} \left(\frac{5}{6}\right)^{i-1} \\ &= 1/(1 - 5/6) = 6 < \infty \end{aligned}$$

and R is a stopping time with finite expectation for the process $\{Z_n : n \geq 1\}$ and thus for the process $\{X_n : n \geq 1\}$ as well

- so with $X_0 = 0$ by Wald's Theorem $E(X_R) = (3.5)\,6 = 21$

- also $S = \inf\{n \geq 1 : Z_n = 3\}$ is a stopping time for this process with expectation 6 so $E(X_S) = 21$

- now consider $E(X_{R-1})$ and note
$\{R-1 = n\} = \{R = n+1\} \notin \mathcal{A}_{X_0,\ldots,X_n}$ so $R-1$ is not a stopping time for the process$\{X_n : n \geq 1\}$

- but $X_{R-1} = X_R - 5$ and so
$E(X_{R-1}) = E(X_R) - 5 = 21 - 5 = 16 < 21 - 3 = E(X_{S-1})$

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- recall Gambler's Ruin where we showed if $T_i =$ 1st time $X_n = i$ we showed (Lecture 3c)

$$s(a) = P_a(T_0 > T_c) = \text{prob. gambler acquires full fortune}$$
$$= \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^a}{1-\left(\frac{1-p}{p}\right)^c} & p \neq 1/2 \\ a/c & p = 1/2 \end{cases}$$
$$r(a) = P_a(T_0 < T_c) = \text{prob. gambler is ruined}$$
$$= \begin{cases} \frac{1-\left(\frac{p}{1-p}\right)^{c-a}}{1-\left(\frac{p}{1-p}\right)^c} & p \neq 1/2 \\ (c-a)/c & p = 1/2 \end{cases}$$

and put $T = \min(T_0, T_c) =$ stopping time of the gambling

Proposition V.12 *(Expected stopping time for Gambler's ruin)* If $\{X_n : n \geq 0\}$ is the random walk with $X_0 = a > 0$ and $Z_1, Z_2, \ldots \overset{i.i.d.}{\sim} 2\text{Bernoulli}(p) - 1$ with $p \neq 1/2$, then with $T = \min(T_0, T_c)$

$$E(T) = \frac{cs(a) - a}{2p - 1}.$$

Proof: We have $E(Z_i) = 2p - 1$. Let $T_n = \min(T, n)$ so $T_n \uparrow T$ and MCT gives $E(T_n) \to E(T)$. So by Wald's Theorem

$$E(X_{T_n}) = a + (2p-1)E(T_n).$$

Now $X_{T_n} \overset{wp1}{\to} X_T$ and since $X_{T_n} \in \{0, 1, \ldots, c\}$ then $|X_{T_n}| \leq c$ and so the DCT implies $E(X_{T_n}) \to E(X_T)$. Together these imply

$$E(X_T) = \lim_{n\to\infty} E(X_{T_n}) = a + (2p-1) \lim_{n\to\infty} E(T_n) = a + (2p-1)E(T)$$

But also $E(X_T) = cs(a) + 0r(a) = cs(a)$ so $a + (2p-1)E(T) = cs(a)$ which implies

$$E(T) = \frac{cs(a) - a}{2p - 1}.$$

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- note a corollary of this is that $E(T) < \infty$ (a simpler proof as opposed to Prop. 1.7.6 in the text)

- consider now the gambler's ruin problem with $p = 1/2$ so $\{X_n : n \geq 0\}$ is a ssrw and a martingale as in this case $2p - 1 = 0$

- we need a few more results about martingales

Proposition V.12 Let $\{X_n : n \geq 0\}$ be a random walk with $X_n = \sum_{i=0}^n Z_i$ where $Z_0 = a$ and $Z_1, Z_2, \ldots$ are $i.i.d.$ with mean 0 and variance $v < \infty$. The the process $\{Y_n : n \geq 0\}$, where $Y_n = (X_n - a)^2 - nv = \left(\sum_{i=1}^n Z_i\right)^2 - nv$, is a martingale.

Proof: We have

$$E\,|Y_n| \leq E\left(\left(\sum_{i=1}^n Z_i\right)^2\right) + nv = nv + nv = 2nv$$

so the process $\{Y_n : n \geq 0\}$ has finite mean. Then, since Z_{n+1} is independent of $Z_0, Z_1, \ldots, Z_n$ and so independent of $Y_0, Y_1, \ldots, Y_n$,

$$
\begin{aligned}
& E(Y_{n+1} \mid Y_0, \ldots, Y_n) \\
= \; & E\left(\left(\sum_{i=1}^{n} Z_i + Z_{n+1} \right)^2 - (n+1)v \mid Y_0, \ldots, Y_n \right) \\
= \; & E\left(\left(\sum_{i=1}^{n} Z_i \right)^2 - nv + 2\left(\sum_{i=1}^{n} Z_i \right) Z_{n+1} + Z_{n+1}^2 - v \mid Y_0, \ldots, Y_n \right) \\
= \; & Y_n + 2E(Z_{n+1})E\left(\left(\sum_{i=1}^{n} Z_i \right) \mid Y_0, \ldots, Y_n \right) + E(Z_{n+1}^2) - v \\
= \; & Y_n + 0 + v - v = Y_n.
\end{aligned}
$$

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Proposition V.13 For the gambler's ruin process $\{X_n : n \geq 0\}$ with $p = 1/2$, then $E(T) = Var(X_T) = a(c-a)$.

Proof: Note first that $Var(Z_i) = p1^2 + (-1)^2(1-p) - (2p-1)^2 = 1 - (2p-1)^2 = 1$ when $p = 1/2$. By Prop. V.12 $Y_n = (X_n - a)^2 - n$ is a martingale. For a constant $m > 0$ then $S_m = \min(T, m)$ is a bounded stopping time for $\{X_n : n \geq 0\}$ and thus $Y_{S_m} = (X_{S_m} - a)^2 - n$ is a r.v.

By the Optional Stopping Lemma (Prop. V.8 slightly extended so if T is a stopping time wrt $\{X_n : n \geq 0\}$ and $\{Y_n : n \geq 0\}$ is a martingale where $Y_n = f(X_n)$ then $E(Y_T) = E(Y_0)$)

$$E(Y_{S_m}) = E(Y_0) = (a-a)^2 - 0 = 0$$

and since $Y_{S_m} = (X_{S_m} - a)^2 - S_m$ this implies

$$E(S_m) = E((X_{S_m} - a)^2).$$

Also, $S_m \uparrow T$ as $m \longrightarrow \infty$ by MCT $E(S_m) \uparrow E(T)$.

Now since $X_{S_m} \in \{0, 1, \ldots, c\}$ then $(X_{S_m} - a)^2 \leq \max\{a^2, (c-a)^2\}$ and $(X_{S_m} - a)^2 \overset{wp1}{\to} (X_T - a)^2$ the DCT implies that

$$E(T) = \lim_{m\to\infty} E(S_m) = \lim_{m\to\infty} E((X_{S_m} - a)^2) \to E((X_T - a)^2) = Var(X_T)$$

since, see Example V.7, $E(X_T) = a$. Finally

$$Var(X_T) = \frac{a}{c}(c-a)^2 + \frac{c-a}{c}a^2 = \frac{a(c-a)}{c}(c-a+a) = a(c-a).$$

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Exercise V.8 Text 3.4.2

Exercise V.9 Suppose $X_1, \ldots, X_n$ are $i.i.d.$ with $E(X_i) = 1$. Prove $Y_n = X_0 X_1 X_2 \cdots X_n$ is a martingale where X_0 is a constant. Such a process is used to model stock prices where X_0 is the initial price and X_i = the rate of return on the stock in period i.

Exercise V.10 A process $\{Y_n : n \geq 0\}$ with $E(Y_n)$ finite for all n, is a *supermartingale (submartingale)* if $E(Y_{n+1} \mid Y_0, \ldots, Y_n) \leq (\geq) Y_n$. Prove that if $\{Y_n : n \geq 0\}$ is a martingale and g is a convex function such that $E(g(Y_n))$ finite for all n, then $\{g(Y_n) : n \geq 0\}$ is a submartingale.

Exercise V.11 If $\{X_n : n \geq 0\}$ is a martingale and $E(X_n^2) < \infty$ for all n then $E((X_{n+1} - X_n)^2 \mid X_0, \ldots, X_n) = E(X_{n+1}^2 \mid X_0, \ldots, X_n) - X_n^2$.

Exercise V.12 If $\{X_n : n \geq 0\}$ is a martingale and $E(X_n^2) < \infty$ for all n and $0 \leq i \leq i \leq k < n$ then

$$\begin{aligned} E((X_n - X_k)X_j) &= 0, \\ E((X_n - X_k)(X_j - X_i)) &= 0. \end{aligned}$$

The last property is referred to as orthogonal increments. Recall that $< X, Y >= E(XY)$ is an inner product on $L^2(\Omega, \mathcal{A}, P) =$ the set of all random variables defined on the probability space with finite second moment. Deduce from this that

$$E((X_n - X_0)^2) = \sum_{k=1}^{n} E((X_k - X_{k-1})^2).$$