Probability and Stochastic Processes II - Lecture 5b

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- Proposition V.8 required that the stopping time T be bounded $wp1$ but more general versions are required as most stopping times aren't bounded as in $T = \inf\{n : a \leq X_n \leq b\}$ when $P(a \leq X_n \leq b) < 1$ for every n

Proposition V.9 *(Optional Stopping Theorem)* If $\{X_n : n \in \mathbb{N}_0\}$ is a martingale with stopping time T satisfying $P(T < \infty) = 1$, such that $E|X_T| < \infty$ and $\lim_{n\to\infty} E(X_n I_{\{T>n\}}) = 0$, then $E(X_T) = E(X_0)$.

Proof: For $m \in \mathbb{N}_0$ put $S_m = \inf\{T, m\}$. Then S_m is a stopping time and it is bounded since $P(S_m \leq m) = 1$. So Prop. V.8 implies $E(X_{S_m}) = E(X_0)$. Now

$$\begin{aligned} X_{S_m} &= X_T I_{T\leq m} + X_m I_{T>m} = X_T(1 - I_{T>m}) + X_m I_{T>m} \\ &= X_T - X_T I_{T>m} + X_m I_{T>m} \text{ or} \\ X_T &= X_{S_m} + X_T I_{T>m} - X_m I_{T>m} \end{aligned}$$

which implies

$$E(X_T) = E(X_0) + E(X_T I_{T>m}) - E(X_m I_{T>m})$$

and letting $m \to \infty$ implies

$$
\begin{aligned}
E(X_T) &= E(X_0) + \lim_{m\to\infty} (E(X_T I_{T>m}) - E(X_m I_{T>m})) \\
&= E(X_0) + \lim_{m\to\infty} E(X_T I_{T>m}) \text{ by hypothesis and}
\end{aligned}
$$

since $|X_T I_{T>m}| \le |X_T|$, $E|X_T| < \infty$ and $\lim_{m\to\infty} |X_T I_{T>m}| \overset{wp1}{=} 0$ since $X_{T(\omega)}(\omega) I_{T(\omega)>m}(\omega) \to 0$ for any fixed ω. Therefore, by the DCT we have $\lim_{m\to\infty} E(X_T I_{T>m}) = 0$ and the result is proved. ■

Proposition V.10 *(Optional Stopping Corollary)* If $\{X_n : n \in \mathbb{N}_0\}$ is a martingale with stopping time T satisfying $P(T < \infty) = 1$, and is also *bounded up to time* T (there is m s.t. $P(|X_n| I_{n \le T} \le m) = 1$ for every n), then $E(X_T) = E(X_0)$.

Proof: We have

$$P(|X_T| > m) = \sum_n P(T = n, |X_n| > m)$$

$$\begin{aligned} &= \sum_n P(T = n, |X_n| I_{n \le T} > m) \\ &\le \sum_n P(|X_n| I_{n \le T} > m) = 0. \end{aligned}$$

Therefore, $E|X_T| \le m$ and

$$\begin{aligned} |E(X_n I_{n<T})| &\le E(|X_n| I_{n<T}) = E((|X_n| I_{n \le T}) I_{n<T}) \\ &\le mE(I_{n<T}) = mP(T > n) \to 0 \end{aligned}$$

with n as $\{T > n\} \downarrow \{T = \infty\}$ and $P(T = \infty) = 0$ so $P(T > n) \downarrow 0$ by cty of P. The hypotheses of Prop. V.9 then apply and the result follows. ■

Example V.7 *Gambler's ruin*

- recall a gambler, starting with fortune a, wins 1 unit with probability p and the house has capital $c-a$ and X_n is the fortune of the gamber at time n and $X_n = a + \sum_{i=1}^{n} Z_i$ where the Z_i are iid 2Bernoulli$(p)-1$

- $\{X_n : n \in \mathbb{N}_0\}$ is a srw and a ssrw when $p = 1/2$ and so is a martingale

- if $T_i =$ 1st time $X_n = i$ we showed (Lecture 3c)

$$
\begin{aligned}
s(a) &= P_a(T_c < T_0) = \text{prob. gambler acquires full fortune} \\
&= \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^a}{1-\left(\frac{1-p}{p}\right)^c} & p \neq 1/2 \\ a/c & p = 1/2 \end{cases} \\
r(a) &= P_a(T_0 < T_c) = \text{prob. gambler is ruined} \\
&= \begin{cases} \frac{1-\left(\frac{p}{1-p}\right)^{c-a}}{1-\left(\frac{p}{1-p}\right)^c} & p \neq 1/2 \\ (c-a)/c & p = 1/2 \end{cases}
\end{aligned}
$$

Exercise V.4 Show T_i is a stopping time for each $i \in \{0, 1, \ldots, c\}$.

Exercise V.5 Prove that, if T_1, T_2 are stopping times, then $\min(T_1, T_2)$ is a stopping time. What about $\max(T_1, T_2)$, $T_1 T_2$ and $T_1 + T_2$?

- therefore

$$T = \inf\{n : X_n = 0 \text{ or } X_n = c\} = \min(T_0, T_c)$$

is a a stopping time and (check the algebra)

$$\begin{aligned} & P_a(T < \infty) = P_a(T_c < T_0 \text{ or } T_0 < T_c) \\ = \ & P_a(T_c < T_0) + P_a(T_0 < T_c) = s(a) + r(a) = 1 \end{aligned}$$

- also, since $X_n \in \{0, \ldots, c\}$ then $P(|X_n| I_{n \le T} \le c) = 1$ for every n

- recall $\{X_n : n \in \mathbb{N}_0\}$ is a martingale iff $p = 1/2$ so in that case we have $E(X_T) = E(X_0) = a$

- when $p \neq 1/2$ consider the new process

$$Y_n = \left(\frac{1-p}{p}\right)^{X_n}$$

- since $X_n = \frac{p}{1-p}\log(Y_n)$ and $X_{n+1} = X_n + Z_{n+1}$

$$E(Y_{n+1} \mid Y_0, \ldots, Y_n) = E\left(\left(\frac{1-p}{p}\right)^{X_{n+1}} \mid X_0, \ldots, X_n\right)$$

$$= \left(\frac{1-p}{p}\right)^{X_n} E\left(\left(\frac{1-p}{p}\right)^{Z_{n+1}}\right)$$

$$= \left(\frac{1-p}{p}\right)^{X_n} \left[(1-p)\left(\frac{1-p}{p}\right)^{-1} + p\left(\frac{1-p}{p}\right)^{1}\right]$$

$$= \left(\frac{1-p}{p}\right)^{X_n} = Y_n$$

and so $\{Y_n : n \in \mathbb{N}_0\}$ is a martingale

- also $\mathcal{A}_{X_0,\ldots,X_n} = \mathcal{A}_{Y_0,\ldots,Y_n}$ we have that T is a stopping time for $\{Y_n : n \in \mathbb{N}_0\}$ with $P(T < \infty) = 1$

- finally, with $m = \max\left\{1, \left(\frac{1-p}{p}\right)^c\right\}$

$$P(|Y_n| I_{n \le T} \le m) = P\left(\left(\frac{1-p}{p}\right)^{X_n} I_{n \le T} \le m\right) = 1$$

because

$$\begin{aligned} \left(\frac{1-p}{p}\right)^{X_n} I_{n \le T} &\le m = \left(\frac{1-p}{p}\right)^c \text{ when } p < 1/2 \\ \left(\frac{1-p}{p}\right)^{X_n} I_{n \le T} &\le m = 1 \text{ when } p > 1/2 \end{aligned}$$

- therefore by Prop. V.10

$$E(Y_T) = E(Y_0) = \left(\frac{1-p}{p}\right)^a$$

■

Exercise V.6 Text 3.2.10

Exercise V.7 Text 3.2.11