Probability and Stochastic Processes II - Lecture 3c

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III.4 Simple Random Walk with Absorbing Barriers (Gambler's Ruin)

- consider $Z_1, Z_2, \ldots \overset{i.i.d.}{\sim} 2\text{Bernoulli}(p) - 1$, so $P(Z_i = 1) = p, P(Z_i = -1) = 1 - p$, and independent of Z_0

- put $X_n = Z_0 + \sum_{i=1}^{n} Z_i$

- we proved, when $Z_0 \equiv i$, that

$$p_{ij}^{(n)} = P\left(\sum_{k=1}^{n} Z_k = j - i\right)$$

$$= \begin{cases} 0 & \text{if } n + j - i \text{ not even} \\ \binom{n}{\frac{n+j-i}{2}} p^{\frac{n+j-i}{2}} (1-p)^{n - \frac{n+j-i}{2}} & \text{if } n + j - i \in \{0, 2, \ldots, 2n\} \end{cases}$$

- now suppose $Z_0 \equiv a \in \mathbb{N}_0$, a gambler's initial fortune, and the house has fortune $c - a$ so the total fortune at stake is c

- so for $j \in \{0, \ldots, c\}$

$$p_{aj}^{(n)} = P\left(\sum_{k=1}^{n} Z_k = j - a\right)$$

$$= \begin{cases} 0 & \text{if } n + j - a \text{ not even} \\ \binom{n}{\frac{n+j-a}{2}} p^{\frac{n+j-a}{2}} (1-p)^{\frac{n-j+a}{2}} & \text{if } n + j - a \in \{0, 2, \ldots, 2n\} \end{cases}$$

- at each time the gamber bets and wins one 1 unit with probability p and loses one unit to the house with probability $1 - p$

- the gambling ends when $X_n = 0$ or $X_n = c$

- let $T_i = $ 1st time $X_n = i$

- compute

$$\begin{aligned} s(a) &= P_a(T_c < T_0) = \text{ prob. gambler acquires entire fortune} \\ &= P_a(X_n = c, \text{ for some } n \text{ and } X_1, \ldots, X_{n-1} \neq 0) \end{aligned}$$

- note that the function s satisfies the boundary conditions $s(0) = 0, s(c) = 1$

- s also satisfies the difference equation for $a \in \{1, \ldots, c-1\}$

$$
\begin{aligned}
s(a) &= P_a(T_c < T_0) \overset{TTP}{=} \begin{array}{c} P_a(T_c < T_0 \,|\, X_1 = a+1)p+ \\ P_a(T_c < T_0 \,|\, X_1 = a-1)(1-p) \end{array} \\
&\overset{MP}{=} ps(a+1) + (1-p)s(a-1) \text{ so using } s(a) = ps(a) + (1-p)s(a) \\
& p(s(a+1) - s(a)) = (1-p)(s(a) - s(a-1)) \text{ or} \\
& s(a+1) - s(a) = \left(\frac{1-p}{p}\right)(s(a) - s(a-1)) \\
&= \ldots = \left(\frac{1-p}{p}\right)^a (s(1) - s(0)) = \left(\frac{1-p}{p}\right)^a s(1) \text{ so} \\
& s(a) = (s(a) - s(a-1)) + (s(a-1) - s(a-2)) + \cdots + (s(1) - s(0 \\
&= \sum_{k=1}^{a} \left(\frac{1-p}{p}\right)^{k-1} s(1) = \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^a}{1-\left(\frac{1-p}{p}\right)} s(1) & p \neq 1/2 \\ as(1) & p = 1/2 \end{cases} \quad \text{so}
\end{aligned}
$$

$$
\begin{aligned}
1 &= s(c) = \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^c}{1-\left(\frac{1-p}{p}\right)} s(1) & p \neq 1/2 \\ as(1) & p = 1/2 \end{cases} \quad \text{or} \\
s(1) &= \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)}{1-\left(\frac{1-p}{p}\right)^c} & p \neq 1/2 \\ 1/c & p = 1/2 \end{cases} \quad \text{which implies} \\
s(a) &= \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^a}{1-\left(\frac{1-p}{p}\right)^c} & p \neq 1/2 \\ a/c & p = 1/2 \end{cases}
\end{aligned}
$$

- Gambler's ruin probability (reverse role of p and $1-p$ and a and $c-a$)

$$
\begin{aligned}
r(a) = P_a(T_0 < T_c) &= \text{prob. gambler is ruined} \\
&= \begin{cases} \frac{1-\left(\frac{p}{1-p}\right)^{c-a}}{1-\left(\frac{p}{1-p}\right)^c} & p \neq 1/2 \\ (c-a)/c & p = 1/2 \end{cases}
\end{aligned}
$$

- typically $c - a >> a$ and we want $\lim_{c\to\infty} P(T_0 < T_c)$
- $\{T_0 < T_c\}$ is monotone increasing in c and since $T_c \geq c - a$ then

$$\{T_0 < c - a\} \subset \{T_0 < T_c\} \subset \{T_0 < \infty\}$$

which implies (using convergence of events)

$$\{T_0 < \infty\} = \lim_{c\to\infty}\{T_0 < c - a\} \leq \lim_{c\to\infty}\{T_0 < T_c\} \leq \{T_0 < \infty\}$$

and so, by continuity of the probability measure,
$P(T_0 < \infty) = \lim_{c\to\infty} P(T_0 < T_c)$ implying

$$P(T_0 < \infty) = \lim_{c\to\infty} P(T_0 < T_c) = \begin{cases} 1 & p \leq 1/2 \\ \left(\frac{1-p}{p}\right)^a & p > 1/2 \end{cases}$$

$$P(T_0 = \infty) = \begin{cases} 0 & p \leq 1/2 \\ 1 - \left(\frac{1-p}{p}\right)^a & p > 1/2 \end{cases}$$

- so in practical circumstances ruin is certain but if the gambler has an edge $p > 1/2$, then the probability they are never ruined is $1 - \left(\frac{1-p}{p}\right)^a$

Exercises

Exercise III.4.1 Text 1.7.2.

Exercise III.4.2 (Text 1.7.4) Prove $r(a) + s(a) = 1$ so the game ends with probability 1. So if $T = \min\{T_0, T_c\}$ = time game ends and this is finite with probability 1. Propostion 1.7.6 also shows $E(T) < \infty$.

Exercise III.4.3 Text 1.7.9

Exercise III.4.4 Text 1.7.11