Probability and Stochastic Processes II - Lecture 3b

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III.2 Recurrence and Transience

- for MC $\{X_n : n \in \mathbb{N}_0\}$ with state space $\mathcal{S} \subset \mathbb{Z}$ let $P_i = P(\cdot \mid X_0 = i)$ (so $p_{ij}^{(n)} = P_i(X_n = j)$) and let E_i denote expectation wrt P_i

Definition III.2.1 The (i,j)*-th return probability* for the MC $\{X_n : n \in \mathbb{N}_0\}$ is given by

$$\begin{aligned} f_{ij} &= P_i(X_n = j \text{ for some } n) = \sum_{n=1}^{\infty} P_i(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = j) \\ &= \text{the probability the chain visits state } j \text{ after starting in state } i \end{aligned}$$

- so

$$f_{ii} = \text{the prob. the chain returns to state } i \text{ after starting in state } i$$

$$1 - f_{ij} = \text{the prob. the chain never hits state } j \text{ after starting in state } i$$

- note

$$\begin{aligned}
& P_i(X_n = j \text{ and } X_{n+m} = k \text{ for some } n \text{ and } m) \\
& \text{and by TTP} \\
= & \sum_{n=1}^{\infty} P(X_{n+m} = k \text{ for some } m \,|\, X_0 = i, X_1 \neq j, \ldots X_{n-1} \neq j, X_n = j) \times \\
& P_i(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = j) \text{ and by Prop. III.3 and TH} \\
= & \sum_{n=1}^{\infty} P(X_m = k \text{ for some } m \,|\, X_0 = j) P_i(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = j) \\
= & f_{jk} \sum_{n=1}^{\infty} P_i(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = j) = f_{ij} f_{jk}
\end{aligned}$$

- let $N_i = \#\{n : X_n = i\} \in \mathbb{N}_0 \cup \{\infty\}$ and N_i is the number of visits the chain makes to state i

- therefore the conditional distribution of N_j given $X_0 = i$ is given by (using argument on previous slide)

$$\begin{aligned}
P_i(N_j &= 0) = 1 - f_{ij} \\
P_i(N_j &\geq 1) = f_{ij}, P_i(N_j \geq 2) = f_{ij} f_{jj}, \ldots, P_i(N_j \geq k) = f_{ij} f_{jj}^{k-1} \\
P_i(N_j &= k) = P_i(N_j \geq k) - P_i(N_j \geq k+1) = f_{ij} f_{jj}^{k-1}(1 - f_{jj}) \\
P_i(N_i &= k) = f_{ii}^k (1 - f_{ii}) \ (= 0 \text{ when } f_{ii} = 1) \\
P_i(N_i &= \infty) = 1 - \sum_{k=0}^{\infty} P_i(N_i = k) = \begin{cases} 1 & f_{ii} = 1 \\ 0 & f_{ii} < 1 \end{cases}
\end{aligned}$$

Definition III.2.1 A state i of a MC is *recurrent* if $f_{ii} = 1$ and is *transient* if $f_{ii} < 1$.

Proposition III.6 *(Recurrent state theorem)* For a MC
(i) state i is recurrent iff $P_i(N_i = \infty) = 1$ iff $E_i(N_i) = \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$,
(ii) state i is transient iff $P_i(N_i = \infty) = 0$ iff $E_i(N_i) = \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

Proof: The first parts of (i) and (ii) follow from the conditional distribution of N_i given $X_0 = i$ just worked out. Now note that $N_i = \sum_{n=1}^{\infty} I_{\{i\}}(X_n)$ and so

$$E_i(N_i) = E_i\left(\sum_{n=1}^{\infty} I_{\{i\}}(X_n)\right) \overset{MCT}{=} \sum_{n=1}^{\infty} E_i(I_{\{i\}}(X_n)) = \sum_{n=1}^{\infty} p_{ii}^{(n)}$$

and when i is recurrent $P_i(N_i = \infty) = 1$ so $E_i(N_i) = \infty$. Now when $f_{ii} < 1$, then

$$E_i(N_i) = \sum_{k=1}^{\infty} k f_{ii}^k (1 - f_{ii}) = \left(f_{ii} - f_{ii}^2\right) + 2\left(f_{ii}^2 - f_{ii}^3\right) + \ldots$$

$$= \sum_{k=1}^{\infty} f_{ii}^k = \left(\frac{1}{1 - f_{ii}} - 1\right) = \frac{f_{ii}}{1 - f_{ii}} < \infty.$$

So when $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ then state i is recurrent and is otherwise transient. ■

Side note - an alternative expression for the expectation of a r. v.

Lemma If $X \geq 0$ then $E(X) = \int_0^\infty (1 - F_X(z))\,dz$ and when $P(X \in \mathbb{N}_0) = 1$ this equals $E(X) = \sum_{k=1}^\infty P(X \geq k)$.

Proof: We have

$$
\begin{aligned}
E(X) &= \int_0^\infty x P_X(dx) = \int_0^\infty \int_0^x dz\, P_X(dx) \\
&= \int_0^\infty \int_0^\infty I_{(0,x)}(z) dz\, P_X(dx) \overset{Fubini}{=} \int_0^\infty \int_0^\infty I_{(0,x)}(z) P_X(dx)\, dz \\
&= \int_0^\infty P(X > z)\, dz = \int_0^\infty (1 - F_X(z))\, dz \\
&= \sum_{k=1}^\infty P(X \geq k) \text{ when } P(X \in \mathbb{N}_0) = 1. \blacksquare
\end{aligned}
$$

Exercise III.2.1 If $E(X)$ exists, then prove

$$
\begin{aligned}
E(X) &= \int_0^\infty (1 - F_X(z))\, dz - \int_{-\infty}^0 F_X(z)\, dz \\
&= \sum_{k=1}^\infty P(X \geq k) - \sum_{k=-1}^{-\infty} P(X \leq k) \text{ when } P(X \in \mathbb{N}_0) = 1.
\end{aligned}
$$

Example III.2.1 *Simple random walk*

- we showed (assuming $X_0 = 0$)

$$p_{ij}^{(n)} = \begin{cases} 0 & \text{if } n+j-i \text{ not even} \\ \binom{n}{\frac{n+j-i}{2}} p^{\frac{n+j-i}{2}} (1-p)^{n-\frac{n+j-i}{2}} & \text{if } n+j-i \in \{0, 2, \ldots, 2n\} \end{cases}$$

- therefore

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty} \binom{2n}{n} p^n (1-p)^n = \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} p^n (1-p)^n$$

- Stirling's approximation to $n!$ says, putting $s_n = (n/e)^n \sqrt{2\pi n}$, that $\lim_{n\to\infty} n!/s_n = 1$ and so

$$\lim_{n\to\infty} \frac{(2n)!}{n!n!} \frac{s_n^2}{s_{2n}} = 1 \text{ and}$$

$$t_n = \frac{s_{2n}}{s_n^2} = \frac{2^{2n}(n/e)^{2n}\sqrt{4\pi n}}{(n/e)^{2n}(2\pi n)} = \frac{2^{2n}}{\sqrt{\pi n}}$$

- now

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}p^n(1-p)^n &= \sum_{n=1}^{\infty}\left(\frac{(2n)!}{n!n!t_n}\right)t_np^n(1-p)^n\\ &= \sum_{n=1}^{\infty}\left(\frac{(2n)!}{n!n!t_n}\right)\frac{(4p(1-p))^n}{\sqrt{\pi n}}\end{aligned}$$

and note that $0 \leq 4p(1-p) \leq 1$ is maximized at $p = 1/2$ and equals 1 there

- for large n the n-th term in this series is like $\frac{(4p(1-p))^n}{\sqrt{\pi n}}$ and

$$\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{\sqrt{\pi n}}\begin{cases} <\infty & p\neq 1/2\\ =\infty & p=1/2\end{cases}$$

and

$$0 < b \leq \left(\frac{(2n)!}{n!n!t_n}\right) \leq B \text{ for all } n \text{ for some } (b,B) \text{ which implies}$$

$$\sum_{n=1}^{\infty}p_{ii}^{(n)} = \begin{cases} <\infty & p\neq 1/2\\ =\infty & p=1/2\end{cases}$$

and so any i is recurrent for a symmetric srw and in the nonsymmetric case all states i are transient ■

Exercises

III.2.2 Text 1.5.12

III.2.3 Text 1.5.15

III.3 Communicating States and Irreducibility

Definition III.3.1 State i *communicates* with state j if $f_{ij} > 0$ denoted by $i \to j$ (equivalently there exist n s.t. $p_{ij}^{(n)} > 0$). A MC is *irreducible* if $i \to j$ for all $i, j \in \mathcal{S}$.

Lemma III.7 (*Sum Lemma*) If $i \to k$ and $l \to j$ and $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$, then $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$. In particular, if $i \leftrightarrow k$ and $\sum_{n=1}^{\infty} p_{kk}^{(n)} = \infty$, then $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ so if i and k mutually communicate and one is recurrent then both are recurrent.

Proof: There exist $m, r \geq 1$ s.t. $p_{ik}^{(m)} > 0, p_{lj}^{(r)} > 0$. The Chapman-Kolmogorov inequality implies $p_{ij}^{(m+s+r)} \geq p_{ik}^{(m)} p_{kl}^{(s)} p_{lj}^{(r)}$ for all s. Therefore,

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} \geq \sum_{s=1}^{\infty} p_{ij}^{(m+s+r)} \geq \sum_{s=1}^{\infty} p_{ik}^{(m)} p_{kl}^{(s)} p_{lj}^{(r)} = p_{ik}^{(m)} p_{lj}^{(r)} \sum_{s=1}^{\infty} p_{kl}^{(s)} = \infty$$

as required. Now suppose $i \leftrightarrow k$ and put $j = i, l = k$ which implies $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ whenever $\sum_{n=1}^{\infty} p_{kk}^{(n)} = \infty$ and conversely. ■

Proposition III.8 *(Cases theorem)* For an irreducible MC either (i) $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all i, j and so all states are recurrent or (ii) $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$ for all i, j and all states are transient.

Proof: (i) If $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$ for some k, l, then by Lemma **III.7** $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all i, j which implies $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ for all i. (ii) If $\sum_{n=1}^{\infty} p_{kl}^{(n)} < \infty$ for all k, l then $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ for all i. ■

Example III.3.1 *Simple random walk*

- we have

$$p_{ij}^{(n)} = \begin{cases} 0 & \text{if } n + j - i \text{ not even} \\ \binom{n}{\frac{n+j-i}{2}} p^{\frac{n+j-i}{2}} (1-p)^{n-\frac{n+j-i}{2}} & \text{if } n + j - i \in \{0, 2, \ldots, 2n\} \end{cases}$$

so choosing n s.t. $n + j - i$ is even we have that $p_{ij}^{(n)} > 0$ so $i \to j$ and similarly $j \to i$ and a srw is irreducible ■

Proposition III.9 *(Finite State Space Theorem)* If the MC is irreducible and S is finite then the chain is recurrent.

Proof: We have

$$\sum_{j\in\mathcal{S}}\sum_{n=1}^{\infty} p_{ij}^{(n)} \overset{Fubini}{=} \sum_{n=1}^{\infty}\sum_{j\in\mathcal{S}} p_{ij}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty$$

and since $\sum_{j\in\mathcal{S}}$ is a finite sum this implies $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for some j and result follows. ■

- put

$$\begin{aligned} H_{ij} &= \{\text{for some } n,\ X_1 \neq j, \ldots X_{n-1} \neq j, X_n = i\} \\ &= \text{ event chain hits } i \text{ before } j \end{aligned}$$

Lemma III.10 *(Hit Lemma)* If $j \to i$ with $j \neq i$ then $P_j(H_{ij}) > 0$.

Proof: Since $j \to i$ there exists a sequence of states $i_0, i_1, \ldots, i_m$ with $i_0 = j, i_m = i$ and such that $p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{m-1} i_m} > 0$. Then there is a shortest such subsequence within this sequence which starts with j, and since this chain is in H_{ij} this proves the result. ■

Lemma III.11 *(f-Lemma)* If $j \to i$ and $f_{jj} = 1$, then $f_{ij} = 1$.

Proof: This is clear when $i = j$ so assume $i \neq j$. Then $P_j(H_{ij}) > 0$. Now

$$\begin{aligned}
& 0 = 1 - f_{jj} = P_j(\text{never return to } j) \\
\geq\ & P_j(\text{for some } n,\ X_1 \neq j, \ldots X_{n-1} \neq j, X_n = i \text{ and } X_{n+m} \neq j\ \forall\ m \geq 1) \\
=\ & \sum_{n=1}^{\infty} P_j(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = i \text{ and } X_{n+m} \neq j\ \forall\ m \geq 1) \\
=\ & \sum_{n=1}^{\infty} P_j(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = i) P(X_{n+m} \neq j\ \forall\ m \geq 1 \,|\, X_n = i) \\
=\ & \sum_{n=1}^{\infty} P_j(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = i) P_i(X_m \neq j\ \forall\ m \geq 1) \text{ (TH)} \\
=\ & (1 - f_{ij}) \sum_{n=1}^{\infty} P_j(X_1 \neq j, \ldots X_{n-1} \neq j, X_n = i) = (1 - f_{ij}) P_j(H_{ij}) \geq 0
\end{aligned}$$

and this implies $f_{ij} = 1$. ■

Lemma III.12 *(Infinite Returns Lemma) For an irreducible MC,*

$$P_i(N_j = \infty) = \begin{cases} 1 & \text{for every } i,j \text{ if chain is recurrent} \\ 0 & \text{for every } i,j \text{ if chain is transient.} \end{cases}$$

Proof: This follows from the earlier derivation

$$P_i(N_j = k) = f_{ij} f_{jj}^{k-1}(1 - f_{jj})$$

since this implies $P_i(N_j = k) = 0$ for every $k \in \mathbb{N}$ when the chain is recurrent since $f_{jj} = 1$. When $f_{jj} < 1$, so the chain is transient, then

$$P_i(N_j = \infty) = \lim_{k\to\infty} P_i(N_j \geq k) = \lim_{k\to\infty} f_{ij} f_{jj}^{k-1} = 0.$$

■

- so there are a number of ways to characterize the recurrence or transience of an irreducible MC and the book presents a nice summary (pages 22-23.)

- a subset $C \subset S$ is called *closed* if for every $i \in C, j \notin C$ then $f_{ij} = 0$ so once we get into C we can never leave

- so we can consider a new chain with state space C once we enter C which may be recurrent or transient.

Exercises

III.3.1 Text 1.6.4

III.3.2 Text 1.6.20

III.3.2 Text 1.6.24