STAC63
Midterm 2026

Name:

Student Number:

Any results established in the class or in the Exercises, appropriately referenced, can be used as part of solving these questions. The test is open book as any notes or books can be used, The only restriction is that no electronic devices (computers, phones, watches, etc.) are allowed.

Solutions

1. (a) (10 marks) Suppose $\mathbf{X} \sim N_3(\mu,\Sigma)$ where

$$\mu = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 14 & 5 & 7 \\ 5 & 2 & 2 \\ 7 & 2 & 5 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}$$

and you are asked to approximate the conditional probability

$$p = P((X_1, X_2) \in (-1,3) \times (2,4) \mid X_3)(3).$$

Discuss fully how you would carry out such an approximation including how you would assess the accuracy of the estimate. You do not have to specify code but make sure you describe each step necessary to implement your approximation in a language such as R (you can reference any of the capabilities of such a language).

We have $\underset{\sim}{X}_1 = \binom{X_1}{X_2} \mid X_3 = x_3 \sim N_2(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_3 - \mu_3),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}')$

$= N_2\left(\binom{1}{2} + \binom{7}{2}(5)^{-1}(3-3),\ \begin{pmatrix} 14 & 5 \\ 5 & 2 \end{pmatrix} - \binom{7}{2}(5)^{-1}(7\ \ 2) \right)$

$= N_2\left(\binom{1}{2},\ \begin{pmatrix} 14 & 5 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 49/5 & 14/5 \\ 14/5 & 4/5 \end{pmatrix} \right) = N_2\left(\binom{1}{2},\ \begin{pmatrix} 21/5 & 11/5 \\ 11/5 & 6/5 \end{pmatrix} \right)$

So we generate $\underset{\sim}{X}_1, \ldots, X_n \overset{iid}{\sim} N_2\left(\binom{1}{2},\ \begin{pmatrix} 21/5 & 11/5 \\ 11/5 & 6/5 \end{pmatrix} \right)$ and estimate p by $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} I_{(-1,3)\times(2,4)}(\underset{\sim}{X}_i)$. Then

$\frac{\sqrt{n}(\hat{p} - p)}{\sqrt{\hat{p}(1-\hat{p})}} \approx N(0,1)$. So $\left[\hat{p} - \frac{3}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})},\ \hat{p} + \frac{3}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})}\right]$

contains the true value of p with virtual certainty and we can quote $\frac{3}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})}$ as a measure of the accuracy of $\hat{p}$.

1. (b) (10 marks) Suppose you want to estimate the integral

$$I = \int_{\mathbb{R}^3} \cos(x_1^2 + x_2^3 + x_3^4) \exp(-(x_1^2 + x_2^2 + x_3^2))\, dx_1\, dx_2\, dx_3.$$

Discuss how you will go about this and how you will assess the error in the estimate.

Note that $\exp(-(x_1^2+x_2^2+x_3^2))$ is proportional to a $N_3\left(\underset{\sim}{0}, \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}\right)$ density. So a good importance sampler for this problem is this density and we generate $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n \sim N_3(\underset{\sim}{0}, \frac{1}{2}I)$ and estimate I by $\hat{I} = \frac{1}{n}\sum_{i=1}^{n} \frac{\cos(\underset{\sim}{X}_i'\underset{\sim}{X}_i)}{(2\pi)^{-3/2}(1/2)^{-3/2}} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$

Then putting $s^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$ we have that

$\frac{\sqrt{n}(\bar{y} - I)}{s} \approx N(0,1)$ and so we can quote the interval $\left[\bar{y} - \frac{3s}{\sqrt{n}}, \bar{y} + \frac{3s}{\sqrt{n}}\right]$ as an assessment of the error in the estimate.

2. (a) (10 marks) Suppose that you can generate from the distribution of $\mathbf{W}_n = \mathbf{X}_n / Y_n$ for every n where $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$ and $Y_n \xrightarrow{d} c > 0$ as $n \to \infty$, where c is a known constant. If your goal is to generate from the distribution of $Z = \mathbf{X}'\mathbf{X}$ explain how you would approximately do this and fully justify your approximate method.

Since $\underset{\sim}{X}_n \xrightarrow{d} \underset{\sim}{X}$ then with $g(\underset{\sim}{x}_n) = \underset{\sim}{x}_n' \underset{\sim}{x}_n$ we have that $g(\underset{\sim}{x}_n) \xrightarrow{d} g(\underset{\sim}{x})$ by Prop. IV.1 since g is continuous. Now by Slutsky's Theorem $\underset{\sim}{W}_n = \underset{\sim}{X}_n / Y_n \xrightarrow{d} \underset{\sim}{X} / c$ and so $c^2 \underset{\sim}{W}_n' \underset{\sim}{W}_n \xrightarrow{d} \underset{\sim}{X}' \underset{\sim}{X}$.

So to approximately generate from the distribution of Z we choose n large, generate $\underset{\sim}{W}_n$ and record $c^2 \underset{\sim}{W}_n' \underset{\sim}{W}_n$.

2. (b) (10 marks) Consider the sequence of random variables $X_n(\omega) = \sqrt{n} I_{(0,1/n)}(\omega)$ where $\omega \sim U(0,1)$. Show that this sequence converges in mean to $X = 0$ but does not converge to X in mean square.

We have $E(|X_n - 0|) = \int_0^1 \sqrt{n}\, I_{(0,1/n)}(\omega)\, d\omega$

$= \sqrt{n} \int_0^{1/n} d\omega = \sqrt{n}/n = 1/\sqrt{n} \to 0$ as

$n \to \infty$ so $X_n \xrightarrow{1} 0$. But

$E((X_n - 0)^2) = E(X_n^2) = \int_0^1 n\, I_{(0,1/n)}(\omega)\, d\omega$

$= n \int_0^{1/n} d\omega = n/n = 1 \not\to 0$ so

$X_n \overset{2}{\not\to} 0$.

2. (c). (10 marks) Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} N(4,1)$. Determine an approximation to the distribution of $g(\bar{X}) = \bar{X}^3 + \bar{X}^2 + \bar{X}$ and specifically approximate $P(g(\bar{X}) \leq 1)$.

We have by the CLT that $\sqrt{n}(\bar{X}-4) \xrightarrow{d} N(0,1)$. Now $\frac{dg(x)}{dx} = 3x^2+2x+1$

and $\left.\frac{dg(x)}{dx}\right|_{x=4} = 48+8+1 = 57$ and so by

the Delta Theorem $\sqrt{n}\,(g(\bar{X})-g(4))$ | using $g(4)=84$

$= \sqrt{n}\,(\bar{X}^3+\bar{X}^2+\bar{X}-84) \xrightarrow{d} 57\,N(0,1) = N(0,57^2)$.

Therefore, $P(g(\bar{X}) \leq 1) = P\left(\sqrt{n}\,(g(\bar{X})-84) \leq \sqrt{n}\,(1-84)\right)$

$\approx \Phi\left(-\sqrt{n}\,\frac{83}{57}\right)$.

4. Suppose a fair six-sided die is repeatedly rolled, at times $0, 1, 2, 3, \ldots$ (So, each roll is independently equally likely to be $1, 2, 3, 4, 5$, or 6.) Let $X_0 = 0$, and for $n \geq 1$ let X_n be the largest value that appears among all of the rolls up to time n.

4. (a) (5 marks) Find (with justification) a state space $\mathcal{S}$, initial probabilities $\{\nu_i\}$ and transition probabilities p_{ij}, for which $\{X_n\}$ is Markov chain.

$S = \{0,1,2,3,4,5,6\}$ with initial distribution $\nu(0)=1$ and $P_{ij} = \begin{cases} 0 & \text{when } j<i \\ i/6 & \text{when } j=i \\ 1/6 & \text{when } j>i \end{cases}$ so

since the state only changes from i when a value $j > i$ is obtained. So

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

is the transition matrix

4. (b) (5 marks) Compute the two-step transitions $p_{35}^{(2)}$ and $p_{15}^{(2)}$.

$$P_{35}^{(2)} = P_{33}P_{35} + P_{34}P_{45} + P_{36}P_{65}$$

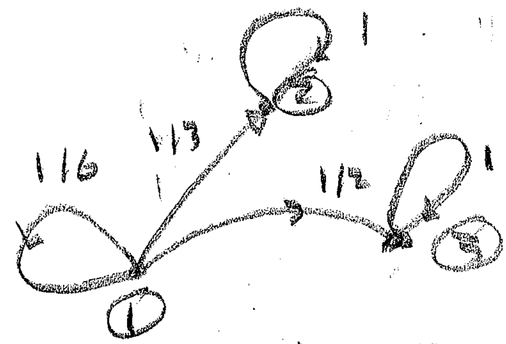
$$= \frac{3}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{5}{6} = \frac{9}{36}.$$

$$P_{15}^{(2)} = P_{11}P_{15} + P_{12}P_{25} + P_{13}P_{35} + P_{14}P_{45} + P_{16}P_{65}$$

$$= \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}1 = \frac{9}{36}.$$

5. Consider a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, and transition probabilities $p_{11} = 1/6, p_{12} = 1/3, p_{13} = 1/2, p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

5. (5 marks) (a) Draw a diagram of this Markov chain.



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

5. (5 marks) (b) Compute (with explanation) f_{12}.

As soon as the transition is made from 1 to 2 so

$$f_{12} = \sum_{n=0}^{\infty} P\left(\begin{array}{c}\text{remaining at 1 for } n \text{ steps and then} \\ \text{transitions to 2}\end{array}\right)$$

$$= \sum_{n=0}^{\infty} p_{11}^{n} p_{12} = \frac{1}{1-p_{11}} p_{12} = \frac{6}{5}\frac{1}{3} = \frac{2}{5}$$

5. (5 marks) (c) Prove that $p_{12}^{(n)} \geq 1/3$, for all positive integers n.

$$P_{12}^{(n)} = \sum_{k=0}^{n-1} P(\text{remains at 1 for } k \text{ transitions and then transitions to 2})$$

$$= \sum_{k=0}^{n-1} P_{11}^{k} P_{12} = P_{12} + \sum_{k=1}^{n-1} P_{11}^{k} P_{12} \geq P_{12} = \frac{1}{3}$$

5. (5 marks) (d) Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

$$p_{12}^{(n)} = p_{12} \sum_{k=0}^{n-1} p_{11}^{k} = p_{12} \frac{1-p_{11}^{n}}{1-p_{11}} = \frac{2}{3}\frac{6}{5}(1-p_{11}^{n})$$

so $\sum_{n=0}^{\infty} p_{12}^{(n)} = \infty$ because $\sum_{n=0}^{\infty} (1-p_{11}^{n}) = \infty$ since

$1-p_{11}^{n} \not\to 0$,

6. Consider a MC with state space $S = \{1, 2, 3\}$ and transition probabilities

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

6. (a) (5 marks) Find a stationary distribution $\{\pi_i\}$ for this chain.

① $12\pi_1 = 4\pi_2 + 3\pi_3$

② $12\pi_2 = 6\pi_1 + 9\pi_3$

③ $12\pi_3 = 6\pi_1 + 8\pi_2$

From ② and ③ $12(\pi_2 - \pi_3) = 9\pi_3 - 8\pi_2$ so

$20\pi_2 = 21\pi_3$ so $\pi_2 = \frac{21}{20}\pi_3$

From ① $12\pi_1 = \left(\frac{21}{5} + 3\right)\pi_3 = \frac{36}{5}\pi_3$

so $\pi_1 = \frac{36}{60}\pi_3 = \frac{3}{5}\pi_3$

$1 = \pi_1 + \pi_2 + \pi_3 = \left(\frac{36 + 63 + 60}{60}\right)\pi_3 = \frac{159}{60}\pi_3 = \frac{53}{20}\pi_3$

So $\pi_3 = \frac{20}{53}$, $\pi_2 = \frac{21}{20}\frac{20}{53} = \frac{21}{53}$, $\pi_1 = \frac{3}{5}\frac{20}{53} = \frac{12}{53}$

6. (b) (5 marks) Determine (with explanation) whether or not $\lim_{n\to\infty} p_{13}^{(n)} = \pi_3$.

Note that $1 \to 2$, $1 \to 3$, $2 \to 1$, $2 \to 3$, $3 \to 1$, $3 \to 2$
so the chain is irreducible and so all states have the same period.

Since $1 \to 2 \to 1$, $1 \to 2 \to 3 \to 1$, then $T(1) = \gcd\{2, 3, \ldots\} = 1$
which implies the chain is aperiodic.

Then by Proposition $\underline{\text{IV}}$.9 $\lim_{n\to\infty} P_{13}^{(n)} = \pi_3$.

6. (c) (5 marks) Determine (with explanation) whether or not $f_{13} = 1$.

By Prop. $\overline{\underline{\text{III}}}$.9 the chain is recurrent and so $f_{11} = f_{22} = f_{33} = 1$. Since $1 \to 3$ by Lemma $\overline{\underline{\text{III}}}$.11 $f_{13} = 1$.

6. (d) (5 marks) Determine (with explanation) whether or not $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$.

By Prop. III.8 (i) $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$.