

# STAC63 Final Exam 2024

Name of the instructor: Michael Evans

Date: April 17, 2024

Start time: 19:00

Duration of the exam: 3 hours

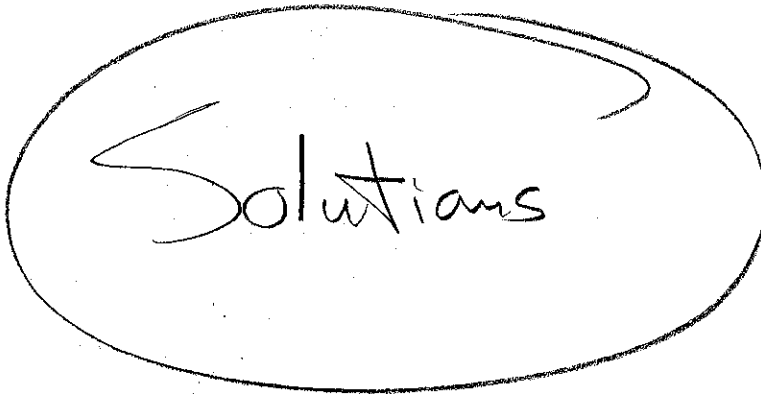
Number of pages of the exam: ? pages (including cover page)

Location of the exam: IC200

Any results established in the class or in the Exercises, appropriately referenced, can be used as part of solving these questions. The exam is open book, so any aids, such as books or notes, are allowed but the use of computers is not permitted.

Name:

Student Number:



Solutions

1. (10 marks) Suppose you are asked to provide an algorithm to generate from the standard logistic distribution with density given by  $f(x) = e^{-x}/(1+e^{-x})^2$  for  $x \in \mathbb{R}^1$ . Assuming you have a source of iid uniform(0, 1) random variables, provide such an algorithm.

$$\text{The cdf is given by } F(x) = \int_{-\infty}^x \frac{e^{-z}}{(1+e^{-z})^2} dz$$

$$= \left. (1+e^{-z})^{-1} \right|_{-\infty}^x = \frac{1}{1+e^{-x}} \quad \text{So}$$

$$u = \frac{1}{1+e^{-x}} \quad \text{iff} \quad \frac{1}{1+e^{-x}} - 1 = e^{-x}$$

$$\text{iff} \quad \frac{1-u}{u} = e^{-x} \quad \text{iff} \quad x = \log\left(\frac{1-u}{u}\right)$$

Therefore  $F^{-1}(u) = \log\left(\frac{u}{1-u}\right)$  and we

generate  $X \sim$  standard logistic by generating  $U \sim U(0,1)$  and putting  $X = \log\left(\frac{U}{1-U}\right)$ .

2 (10 marks) Suppose that the r.v.'s  $X_n \xrightarrow{wp1} X$  where  $X \sim N(0,1)$ . What do  $P(|X_n - X| > 1)$  and  $P(X_n > 1)$  converge to? Justify your conclusions.

Since convergence w.p.1 implies convergence in probability we have  $\lim_{n \rightarrow \infty} P(|X_n - X| > 1) = 0$ .

Also, convergence w.p.1 implies convergence in distribution so  $X_n \xrightarrow{d} X$  which implies  $P(X_n > 1) \rightarrow P(1 = \Phi)$  where  $\Phi$  is the cdf of the  $N(0,1)$  distribution and 1 is a continuity point of  $\Phi$ .

3. Consider the Markov chain  $\{X_n : n = 0, 1, 2, \dots\}$  with states space  $S = \{1, 2, 3\}$ , initial distribution  $\nu = (1/3, 1/3, 1/3)$  and transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

3 (a) (5 marks) Determine the probability distribution of  $X_3$ .

$$(P(X_3=1), P(X_3=2), P(X_3=3)) = \nu P^3$$

$$\text{and } P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 5/8 & 3/8 & 1/4 \\ 5/8 & 1/4 & 1/8 \end{pmatrix}$$

$$\text{So } (P(X_3=1), P(X_3=2), P(X_3=3))$$

$$= (1/3, 1/3, 1/3) \begin{pmatrix} 1 & 0 & 0 \\ 5/8 & 3/8 & 1/4 \\ 5/8 & 1/4 & 1/8 \end{pmatrix} = (2/3, 5/24, 3/24).$$

3 (b) (5 marks) Is this chain irreducible? Justify your conclusion.

This chain is not irreducible because  
 $1 \rightarrow 2$ .

3. (c) (10 marks) Classify the states as recurrent or transient and determine the return probabilities  $f_{ij}$ .

Clearly  $f_{11} = 1$  so state 1 is recurrent.  
 Since  $2 \rightarrow 3 \rightarrow 1$  we must have  $f_{22} < 1$ ,  $f_{33} < 1$   
 and so states 2 and 3 are transient.

We have  $f_{11} = 1$ ,  $f_{12} = 0$ ,  $f_{13} = 0$

$$f_{21} = P_{22}f_{21} + P_{23}f_{31} = \frac{1}{2}f_{21} + \frac{1}{2}f_{31} \quad \therefore f_{21} = f_{31}$$

$$f_{22} = P_{22} + P_{23}f_{32} = \frac{1}{2} + \frac{1}{2}f_{32}$$

$$f_{23} = P_{23} + P_{22}f_{23} = \frac{1}{2} + \frac{1}{2}f_{23} \quad \therefore f_{23} = 1$$

$$f_{31} = P_{31} + P_{31}f_{21} = \frac{1}{2} + \frac{1}{2}f_{21} = \frac{1}{2} + \frac{1}{2}f_{31} \quad \therefore f_{31} = f_{21} = 1$$

$$f_{32} = P_{31}f_{12} + P_{32} = P_{32} = \frac{1}{2}$$

$$\therefore f_{22} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

$$f_{33} = P_{32}f_{23} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

4. Consider the Markov chain  $\{X_n : n = 0, 1, 2, \dots\}$  with states space  $S = \{1, 2, 3\}$ , initial distribution  $\nu = (1/3, 1/3, 1/3)$  and transition probability matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

4 (a) (5 marks) Determine if this chain is irreducible and aperiodic. Justify your conclusions.

We have that  $1 \rightarrow 2, 1 \rightarrow 3$   
 $2 \rightarrow 1, 2 \rightarrow 3$   
 $3 \rightarrow 1, 3 \rightarrow 2$

so  $1 \rightarrow 2, 1 \rightarrow 3$  and  $2 \rightarrow 3$  so the chain is irreducible.

Also,  $T(1) = \{2, 3, \dots\}$  so  $\gcd T(1) = 1$

$T(2) = T(3) = T(1) = 1$ , so the chain is aperiodic.

4 (b) (5 marks) Determine a stationary distribution  $\pi$  for this chain.

We must have

$$(\pi_1, \pi_2, \pi_3) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/4 & 3/4 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

$$\pi_2/3 + \pi_3/4 = \pi_1 \quad (1)$$

$$\pi_1/2 + 3\pi_3/4 = \pi_2 \quad (2) \text{ or}$$

$$\pi_1/2 + 2\pi_3/3 = \pi_3 \quad (3)$$

and substituting (1) into (2) and (3)

$$\pi_2/6 + \pi_3/8 + 3\pi_3/4 = \pi_2 \quad \text{or} \quad (5/8)\pi_3 = (7/8)\pi_3 \quad \text{or} \quad \pi_2 = \frac{6.7}{5.8}\pi_3 = \frac{21}{20}\pi_3$$

$$\text{and } \pi = \pi_1 + \pi_2 + \pi_3 = (4/3)\pi_2 + (5/4)\pi_3 = \left(\frac{4}{3} \frac{21}{20} + \frac{5}{4}\right)\pi_2 = \frac{53}{20}\pi_2$$

$$\therefore \pi_3 = \frac{20}{53}, \quad \pi_2 = \frac{21}{20}\pi_3 = \frac{21}{53} \quad \text{and} \quad \pi_1 = 1 - \pi_2 - \pi_3 = \frac{12}{53}$$

4 (c) (5 marks) Determine whether or not  $\lim_{n \rightarrow \infty} p_{13}^{(n)} = \pi_3$ .

Since the chain is irreducible, aperiodic and has stationary distribution the Markov Chain Convergence Theorem implies that  $\lim_{n \rightarrow \infty} p_{13}^{(n)} = \pi_3$ .

4 (d) (5 marks) Determine whether or not  $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$ .

For a series to converge to a finite value the  $n$ th term must converge to 0 but here the  $n$ th term converges to  $\frac{20}{53} = \pi_3$  so we must have that  $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$ , since.

This also follows from Prop III.8 (Class Theorem) as all states are recurrent.

4. (e) (5 marks) Determine the mean return times  $m_i$  for each of the states.

We have that the chain has a stationary distribution and so it is a positive recurrent chain (Prop IV.22). Therefore (Prop IV.17)

$$m_1 = 53/12 \text{ transitions}$$

$$m_2 = 53/21 \text{ transitions}$$

$$m_3 = 53/20 \text{ transitions}$$

5. Suppose that

$$X \sim N_3 \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \right)$$

5. (a) (5 marks) Determine  $E(X_1 | X_2)$ .

We have that  $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2 \left( \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \right)$ .

and so  $X_1 | X_2 = x_2 \sim N \left( 1 + \frac{1}{2}(x_2 - 2), 1 - 1/4 \right)$ .

This implies that  $E(X_1 | X_2) = 1 + \frac{1}{2}(X_2 - 2)$ .

5. (b) (5 marks) Determine  $E(E(X_1 | X_2) | X_2, X_3)$  and  $E(E(X_1 | X_2, X_3) | X_2)$ .

We have that  $\mathcal{F}_{X_2} \subseteq \mathcal{F}_{X_2, X_3}$  and

$$E(\cdot | X_2) = E(\cdot | \mathcal{F}_{X_2}), \quad E(\cdot | X_2, X_3) = E(\cdot | \mathcal{F}_{X_2, X_3}).$$

Therefore by the tower property we have that

$$\begin{aligned} E(E(X_1 | X_2) | X_2, X_3) &= E(E(X_1 | X_2, X_3) | X_2) \\ &= E(X_1 | X_2) = 1 + \frac{1}{2}(X_2 - 2). \end{aligned}$$

6. (a) (8 marks) Suppose that  $\{X_n : n = 0, 1, 2, \dots\}$  is a stochastic process with state space  $S = \mathbb{N}_0$ . Justify whether or not each of the following is in general a stopping time for such a process:

(i)  $T = X_{10}$ ,

(ii)  $T = \lfloor E(X_{10}) \rfloor$  where  $\lfloor \cdot \rfloor$  denotes the floor function,

(iii)  $T = \inf \left\{ n : \sum_{i=0}^n X_i > 10 \right\}$ ,

(iv)  $T = \sup \left\{ n : \sum_{i=0}^n X_i > 10 \right\}$ .

(i)  $T$  is not a stopping time because the truth of  $\{T \leq n\}$  can't be determined from  $X_0, \dots, X_n$ , also  $\{T \leq n\} \neq \{X_{10} \leq n\}$  when  $n < 10$ .

(ii)  $T$  is a constant and so  $\{T \leq n\} = \Omega$  for all  $n$ , namely  $\{T \leq n\} = \Omega$  when  $n \geq \lfloor E(X_{10}) \rfloor$  and  $\{T \leq n\} = \emptyset$  when  $n < \lfloor E(X_{10}) \rfloor$  so  $T$  is a stopping time.

(iii)  $\{T \leq n\} = \bigcup_{i=0}^n \left\{ \sum_{j=0}^i X_j > 10 \right\} \in \mathcal{F}_{X_0, \dots, X_n}$  for every  $n$  and so  $T$  is a stopping time.

(iv)  $T$  is not a valid stopping time because it cannot be determined from  $X_0, \dots, X_n$  what  $\sup \left\{ n : \sum_{i=0}^n X_i > 10 \right\}$  is.

6. (b) (7 marks) Suppose that  $X_0 = 1$  and  $X_1, X_2, \dots$  are iid r.v.'s and  $Y_n = X_0 X_1 X_2 \dots X_n$ . Under what conditions is  $\{Y_n : n = 0, 1, 2, \dots\}$  a martingale? When is the process  $\{\log Y_n : n = 0, 1, 2, \dots\}$  a martingale? In the cases where these processes are martingales and the stopping time  $T = \min\{100, \inf\{n : X_n > 10\}\}$  what is the expected value of the stopped random variables  $Y_T$  and  $\log Y_T$ , respectively.

We have that  $\mathbb{E}(Y_{n+1} | Y_0, \dots, Y_n) = \mathbb{E}(Y_n X_{n+1} | Y_0, \dots, Y_n)$   
 $= Y_n \mathbb{E}(X_{n+1})$  since  $X_{n+1}$  is independent of  $Y_0, \dots, Y_n$   
 $= Y_n$  provided  $\mathbb{E}(X_{n+1}) = 1$ . So we  
 require  $\mathbb{E}(X_i) = 1 \quad \forall i$ .

For  $\{\log Y_n : n = 0, 1, 2, \dots\}$  to be a martingale  
 we require that  $P(X_i > 0) = 1$  and that  
 $\mathbb{E}(\log X_i) = 0$ .

Since  $T$  is a bounded stopping time  
 by the Optional Stopping Lemma we have  
 that  $\mathbb{E}(Y_T) = \mathbb{E}(Y_0) = 1$  and  $\mathbb{E}(\log Y_T)$   
 $= \mathbb{E}(\log Y_0) = 0$ .

7. (10 marks) Suppose  $\{B_t : t \geq 0\}$  is Brownian motion and  $\theta \in \mathbb{R}^1$ . Prove that  $\{X_t = \exp(\theta B_t - \theta^2 t/2 : t \geq 0\}$  is a martingale.

Suppose  $0 \leq t_1 \leq \dots \leq t_{n-1} \leq t_n$ . Then

$$E(X_{t_n} | X_{t_1}, \dots, X_{t_{n-1}}) = \exp\left\{-\frac{\theta^2 t_n}{2}\right\} E\left(\exp\{\theta B_{t_n}\} | B_{t_1}, \dots, B_{t_{n-1}}\right)$$

since  $A_{B_{t_n} \rightarrow B_{t_{n-1}}} = A_{X_{t_1}, \dots, X_{t_{n-1}}}$

as  $(X_{t_1}, \dots, X_{t_{n-1}})$  is a 1-1 fn of  $(B_{t_1}, \dots, B_{t_{n-1}})$

$$= \exp\left\{\theta B_{t_{n-1}} - \frac{\theta^2 t_{n-1}}{2}\right\} E\left(\exp\{\theta(B_{t_n} - B_{t_{n-1}})\} | B_{t_1}, \dots, B_{t_{n-1}}\right)$$

$$= X_{t_{n-1}} \exp\left\{-\frac{\theta^2(t_n - t_{n-1})}{2}\right\} E\left(\exp\{\theta(B_{t_n} - B_{t_{n-1}})\} | B_{t_1} - B_0, \dots, B_{t_{n-1}} - B_{t_{n-2}}\right)$$

since  $B_{t_1}, \dots, B_{t_{n-1}}$  is a 1-1 fn of

$$B_{t_1} - B_0, \dots, B_{t_{n-1}} - B_{t_{n-2}}$$

$$= X_{t_{n-1}} \exp\left\{-\frac{\theta^2(t_n - t_{n-1})}{2}\right\} E\left(\exp\{\theta(B_{t_n} - B_{t_{n-1}})\}\right) \text{ by independent increments}$$

and since  $B_{t_n} - B_{t_{n-1}} \sim \sqrt{t_n - t_{n-1}} Z$  where  $Z \sim N(0, 1)$

and  $m_Z(t) = \exp\{t^2/2\}$  (moment generating function of  $Z$ )

$$= X_{t_{n-1}} \exp\left\{-\frac{\theta^2(t_n - t_{n-1})}{2}\right\} \exp\left\{\frac{\theta^2(t_n - t_{n-1})}{2}\right\} = X_{t_{n-1}}$$

and  $\{X_t : t \geq 0\}$  is a martingale.