Recall that when $X$ is a nonnegative r.v., and $i \in \{1, \ldots, n\}, j \in \{1, \ldots, 2^n\}$, in Lecture 13 we defined

$$A_{i,j,n} = \{\omega : (i-1) + (j-1)/2^n \leq X(\omega) < (i-1) + j/2^n\} \in \mathcal{A}$$
$$X_n = \sum_{i=1}^{n} \sum_{j=1}^{2^n} ((i-1) + (j-1)/2^n) I_{A_{i,j,n}} = \sum_{i=1}^{n} \sum_{j=1}^{2^n} a_{i,j} I_{A_{i,j,n}},$$

so $X_n$ is a nonnegative simple function satisfying $X_n(\omega) \leq X(\omega)$.Then when $n \leq n'$,

$$\text{if } X(\omega) \geq n, \text{ then } 0 = X_n(\omega) \leq X_{n'}(\omega),$$
$$\text{if } \omega \in A_{i,j,n}, \text{ then } \omega \in A_{i,j',n'} \text{ for some } j' \text{ and } X_n(\omega) \leq X_{n'}(\omega)$$

and $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in \Omega$. We then defined $E(X) = \lim_{n\longrightarrow\infty} E(X_n)$ which exists because $E(X_n)$ is monotone increasing. We want to be sure that this definition doesn't depend on the particular sequence $\{X_n\}$.

Define

$$X_n^* = \sum_{i=1}^{n} \sum_{j=1}^{2^n} ((i-1) + j)/2^n) I_{A_{i,j,n}} = \sum_{i=1}^{n} \sum_{j=1}^{2^n} b_{i,j} I_{A_{i,j,n}}$$

and note that

$$|E(X_n^*) - E(X_n)| \leq \sum_{i=1}^{n} \sum_{j=1}^{2^n} |b_{i,j} - a_{i,j}| \, P(A_{i,j,n}) \leq 1/2^n.$$

Therefore, since $E(X_n) \to E(X)$, we have that $E(X_n^*) \to E(X)$.

Now suppose $Y = \sum_{k=1}^{m} c_k I_{B_k}$ is a nonnegative simple function with $Y \leq X$. Then, since $\{A_{i,j,n}\}$ forms a partition

$$Y = \sum_{k=1}^{m} c_k I_{B_k} = \sum_{k=1}^{m} c_k \sum_{i=1}^{n} \sum_{j=1}^{2^n} I_{A_{i,j,n} \cap B_k} = \sum_{i=1}^{n} \sum_{j=1}^{2^n} \sum_{k=1}^{m} c_k I_{A_{i,j,n} \cap B_k}$$

Now choose $n$ s.t. $n \geq \max\{b_1, \ldots, b_m\}$. Therefore, whenever $\omega \in A_{i,j,n} \cap B_k$, we have $c_k \leq X(\omega) \leq b_{i,j}$ and so

$$E(Y) \leq E(X_n^*),$$

which implies $E(Y) \leq \lim_{n\to\infty} E(X_n^*) = E(X)$. From the definition we gave in Lecture 13 and this result, we see that we can also define $E(X)$ by

$$E(X) = \sup\{E(Y) : Y \text{ is nonnegative, simple and } Y \leq X\},$$

which is a commonly used definition. The definition we gave in Lecture 13, however, is constructive as we provided a particular sequence $X_n$ of nonnegative, simple functions with $X_n \leq X$ such that $E(X_n)$ converges to this supremum. So, there is no dependence in our definition on the particular sequence.