MULTIVARIATE ASYMPTOTIC MODEL EXPONENTIAL AND LOCATION APPROXIMATIONS

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SUMMARY

We examine the functional form of a statistical model with p dimensional variable and parameter and with large sample or asymptotic structure. It is shown that the functional form can be approximated to third order by a location or exponential model in a first derivative neighbourhood of a data point and to second order on a wide range.

The transformation and exponential families provide a wide range of models in statistics. They also provide basic patterns for statistical inference. Some recent asymptotics uses these patterns to obtain statistical inference in a general asymptotic context (e.g. Fraser & Reid, 1993b) . An asymptotic model with one dimensional variable and parameter can be approximated by a location model or an exponential model to order $O(n^{-3/2})$ in a first derivative neighbourhood and to order $O(n^{-1})$ generally (Abebe et al, 1993; Cakmak et al, 1993). In this paper we show that the same holds with a p dimensional variable and

parameter; this provides general structure for a range of inference methods. 1993July19.

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1. INTRODUCTION

We are concerned with a statistical model that has a p dimensional variable and a p dimensional parameter and is asymptotic as some mathematical parameter n such as initial sample size becomes large. For the one dimensional case various asymptotic results of special interest for inference have been established in Fraser & Reid (1993a), Abebe et al (1993), Cakmak et al (1993); specifically, that the model can be approximated closely by an exponential or location model, and that these approximate models can provide $O(n^{-3/2})$ significance probabilities for the initial model.

Consider a statistical model $f(y;\theta)$ with p dimensional variable and p dimensional parameter and suppose the model is asymptotic as some mathematical parameter $n \to \infty$: that for each θ, y is $O_p(n^{-1/2})$ about the maximum density value $\hat{y}(\theta)$; and that $\ell(\theta; y) = \log f(y;\theta)$ with either argument fixed is $O(n)$ and has a unique maximum. For some background, see DiCiccio, Field, & Fraser (1990), Fraser & Reid (1993a),

To obtain canonical models, we standardize at the observed data point y^0 and at the corresponding maximum likelihood value $\theta^0 = \hat{\theta}(y^0)$ and then reexpress the variable and parameter appropriately.

In the $p = 1$ dimensional case we can obtain the exponential type canonical asymptotic model, with expansion coefficients,

$$\begin{pmatrix} a + (3\alpha_4 - 5\alpha_3^2 - 12c)/24n & -\alpha_3/2n^{1/2} & -\{1 + (\alpha_4 - 2\alpha_3^2 - 5c)/2n\} & \alpha_3/n^{1/2} & (\alpha_4 - 3\alpha_3^2 - 6c)/n \\ 0 & 1 & 0 & 0 & - \\ -1 & 0 & c/n & - & - \\ -\alpha_3/n^{1/2} & 0 & - & - & - \\ -\alpha_4/n & - & - & - & - \end{pmatrix}, \tag{1.1}$$

where $a = -(1/2)\log(2\pi)$. The array records the coefficients $a_{ij} = (\partial^i/\partial\theta^i)(\partial^j/\partial\theta^j)\ell(\theta; y)\Big|_{\theta_0, y_0}$ of the Taylor series expansion of $\ell(\theta; y) = \ln f(y;\theta) = \Sigma a_{ij}(\theta - \theta_0)^i (y - y_0)^j / i!j!$ about an observed data point and its corresponding maximum likelihood estimate. The mathematical parameter c records deviation from exponential model form. The zeros in the second row and second column express the exponential model pattern $\exp\{\theta y - c(\theta)\}f(y)$ where $\log f(y;\theta)$ has only one cross derivative that is not zero (from $y\theta$).

In a parallel way we can obtain the location type canonical asymptotic model with expansion coefficients

$$\begin{pmatrix} a + (3\alpha_4 - 5\alpha_3^2 - 12c)/24n & 0 & -\{1 + (-5c)/2n\} & \alpha_3/n^{1/2} & (-\alpha_4 - 6c)/n \\ 0 & 1 & -\alpha_3/n^{1/2} & \alpha_4/n & - \\ -1 & \alpha_3/n^{1/2} & (-\alpha_4 + c)/n & - & - \\ -\alpha_3/n^{1/2} & \alpha_4/n & - & - & - \\ -\alpha_4/n & - & - & - & - \end{pmatrix} \tag{1.2}$$

where c now records the deviation from location model form. The pattern in the coefficients expresses $(\partial/\partial y + \partial/\partial\theta)^\alpha \ell(\theta;y) = 0$ for a location type model $f(y-\theta)$. For background details, see Fraser & Reid (1993a), Abebe et al (1993), and Cakmak et al (1993).

In this paper we show that the same approximation pattern holds with p dimensional variable and p dimensional parameter. In Section 2 we location-scale standardize a general asymptotic model. The exponential and location approximations are developed in Sections 3 and 4, with some brief comment on the calculation of the corresponding canonical parameters in Section 5. Section 6 provides some general discussion.

2. LOCATION-SCALE STANDARDIZATION

Consider a statistical model $f(y;\theta)$ with p dimensional variable and p dimensional parameter and with the asymptotic properties described in Section 1. Let y^0 be a data point of interest and $\theta^0 = \hat{\theta}(y^0)$ be the corresponding maximum likelihood estimate. We consider a Taylor series expansion of $\ell(\theta;y) = \ln f(y;\theta)$ about (θ^0, y^0):

$$\begin{aligned} \ell(\theta;y) = \ell(\theta_0;y_0) &+ \sum a_i(\theta_i - \theta_i^0) + \sum a^\alpha(y_\alpha - y_\alpha^0) \\ &+ \sum a_{ij}(\theta_i - \theta_i^0)(\theta_j - \theta_j^0)/2! + \sum a_i^\alpha(\theta_i - \theta_i^0)(y_\alpha - y_\alpha^0) \\ &+ \sum a^{\alpha\beta}(y_\alpha - y_\alpha^0)(y_\beta - y_\beta^0)/2! + \ldots \end{aligned} \tag{2.1}$$

where for example $a_{ij}^\alpha = (\partial/\partial\theta_i)(\partial/\partial\theta_j)(\partial/\partial y^\alpha)\ell(\theta;y)\Big|_{\theta^0, y^0}$.

In the various steps towards canonical form we will reexpress the initial variable y and parameter θ with coefficients a in terms of a new variable x and parameter φ with coefficients A; then to keep notation manageable we will again use θ, y a for the next stage. The transformations are thus described algorithmically.

We first recenter at the data value y^0 and maximum likelihood value $\theta^0 = \hat{\theta}(y^0)$,

$$x = y - y^0 \quad , \quad \varphi = \theta - \theta^0 \ .$$

The visible effect on the coefficient array is that the first order derivatives with respect to the θ coordinates are equal to zero:

$$\begin{pmatrix} a & a^{\alpha} & a^{\alpha\beta} & a^{\alpha\beta\gamma} \\ 0 & a_i^{\alpha} & a_i^{\alpha\beta} & \cdots \\ a_{ij} & a_{ij}^{\alpha} & \cdots & \\ a_{ijk} & \cdots & & \end{pmatrix} \tag{2.2}$$

The indices run from 1 to p and summation over repeated indices will be used and implied.

We next rescale the parameter to have identity observed information at the expansion point: let $\theta_i = c_{ij}\varphi$ define a new parameter φ so that $I_{k\ell} = -a_{ij}c_{ik}c_{j\ell}$ is an identity matrix array. We obtain the coefficients

$$\begin{pmatrix} a & a^{\alpha} & a^{\alpha\beta} & a^{\alpha\beta\gamma} \\ 0 & a_i^{\alpha} & a_i^{\alpha\beta} & \cdots \\ -I_{ij} & a_{ij}^{\alpha} & \cdots & \\ a_{ijk} & \cdots & & \end{pmatrix} \tag{2.3}$$

where again we use lower case letters for the coefficients in the new expansion.

We then rescale the variable so that the cross Hessian a_i^{α} becomes an identity array I_i^{α}: let $y_{\alpha} = d_{\alpha}^{\gamma}x_{\gamma}$ define a new variable x so that $I_i^{\gamma} = a_i^{\alpha}d_{\alpha}^{\gamma}$ is an identity array. We obtain the coefficients

$$\begin{pmatrix} a & a^{\alpha} & a^{\alpha\beta} & a^{\alpha\beta\gamma} & a^{\alpha\beta\gamma\delta} \\ 0 & I_i^{\alpha} & a_i^{\alpha\beta} & a_i^{\alpha\beta\gamma} & - \\ -I_{ij} & a_{ij}^{\alpha} & a_{ij}^{\alpha\beta} & - & - \\ -\alpha_{ijk} & a_{ijk}^{\alpha} & - & - & - \\ -\alpha_{ijk\ell} & - & - & - & - \end{pmatrix} \tag{2.4}$$

where elements with three indices are $O(n^{-1/2})$, those with four are $O(n^{-1})$, and the lower right elements that are missing are $O(n^{-3/2})$. This follows the pattern in Fraser & Reid (1993a) and Abebe et al (1993). Also we have used $-\alpha_{ijk}$, $-\alpha_{ijk\ell}$ for the third and fourth derivative terms in the likelihood at the data expansion point: these α's correspond to standardized third and fourth cumulant arrays in the pure exponential case.

In the following sections we will transform towards location and exponential form.

3. EXPONENTIAL MODEL APPROXIMATION

We now consider the location-scale standardized model with coefficients as in (2.4) and further transform the parameter and variable towards the exponential type indicated by (1.1).

With a pure exponential model $\ell(\theta, y) = \theta y - c(\theta) - \ln f(y)$ we have all cross derivatives equal to zero except those that are first order in θ and first order in y. This is the pattern we now seek in applying transformations to the arguments for (2.4).

First we define a new parameter φ to obtain appropriate zeros in the second column:

$$\varphi_\alpha = \theta_\alpha + a^\alpha_{ij}\theta_i\theta_j/2! + a^\alpha_{ijk}\theta_i\theta_j\theta_k/3! \tag{3.1}$$

This changes many coefficients in the array as indicated by the scalar case in Cakmak et al (1993). Then using the original notation for the new coefficients we obtain the array

$$\begin{pmatrix} a & a^\alpha & a^{\alpha\beta} & a^{\alpha\beta\gamma} & a^{\alpha\beta\gamma\delta} \\ 0 & I^\alpha_i & a^{\alpha\beta}_i & a^{\alpha\beta\gamma}_i & - \\ -I_{ij} & 0 & a^{\alpha\beta}_{ij} & - & - \\ -\alpha_{ijk} & 0 & - & - & - \\ -\alpha_{ijk\ell} & - & - & - & - \end{pmatrix} . \tag{3.2}$$

Second we define a new variable x to obtain appropriate zeros in the second row

$$x_i = y_i + a^{\alpha\beta}_i y_{\alpha\beta} y_\alpha y_\beta/2! + a^{\alpha\beta\gamma}_i y_\alpha y_\beta y_\gamma/3! \; . \tag{3.3}$$

The resulting array again using the original notation has the form

$$\begin{pmatrix} a & a_i^\alpha & a^{\alpha\beta} & a^{\alpha\beta\gamma} & a^{\alpha\beta\gamma\delta} \\ 0 & I_i^\alpha & 0 & 0 & - \\ -I_{ij} & 0 & c_{ij}^{\alpha\beta} & 0 & - \\ -\alpha_{ijk} & 0 & - & - & - \\ -\alpha_{ijk\ell} & - & - & - & - \end{pmatrix} . \tag{3.4}$$

If the coefficients $c_{ij}^{\alpha\beta}$ are equal to zero, then we have an exponential model to order $O(n^{-3/2})$ and the elements in the first row are available from formulas in Section 4 of Fraser & Reid (1993a).

More generally we can use the fact that $c_{ij}^{\alpha\beta} = O(n^{-1})$ to derive the adjustment $h(y)$ to the first row elements. For this let $g(y;\theta)$ be the exponential model obtained with $c_{ij}^{\alpha\beta} = 0$ (ibid); then

$$\int \{1 + h(y) + c_{ij}^{\alpha\beta}\theta_i\theta_j y^{\alpha\beta}\} g(y;\theta)dy = 1 \tag{3.5}$$

to order $O(n^{-3/2})$. To calculate with the $O(n^{-1})$ terms $c_{ij}^{\alpha\beta}$, $h(y)$, we need $g(y;\theta)$ only to order $O(n^{-1/2})$, which is normal (θ, I). It follows that $h(y)$ is a unique polynomial of order $O(n^{-1})$ and its coefficients give the corrections to be added to the first row of the expansion of the exponential model. For background see the discussion preceding formula (5) in Cakmak et al.(1993). It follows that an asymptotic model can be approximated to order $O(n^{-1})$ by an exponential model on a compact set for the standardized variable and to order $O(n^{-3/2})$ by an exponential model in a first derivative neighbourhood of the data point, except for a constant of $O(n^{-1})$.

We thus have that the first row in (3.4) is uniquely determined by the third and fourth likelihood cumulants α_{ijk}, $\alpha_{ijk\ell}$, together with the non exponential coefficients $c_{ij}^{\alpha\beta}$. While explicit formulas for these may be possible from Section 4 in Fraser & Reid and the calculations described in the preceding paragraph, it suffices here to have the general characteristics of the coefficients.

We now derive an invariant type expression for the exponential model described by replacing the $c_{ij}^{\alpha\beta}$ by zeros. If we change the expansion point, we find that there is no change in the top left element to order $O(n^{-3/2})$; see the discussions in Cakmak et al (1993). The first column corresponds to the observed likelihood $\ell^0(\theta) = \ell(\theta; y^0)$; the second column corresponds to the gradient of the likelihood

$$\varphi = \varphi(\theta) = (\partial/\partial y')\ell(\theta; y)\Big|_{y^0} = \ell_{;y}(\theta; y^0) \tag{3.6}$$

at the data point; the second row corresponds to the score variable at the observed maximum likelihood estimate, $s = (\partial/\partial y')(\ell(\theta; y)\Big|_{\hat{\theta}^0} = \ell_{\varphi}(\hat{\theta}^0; y)$. The exponential model defined by $c_{ij}^{\alpha\beta} = 0$ is then given by

$$\begin{aligned} &\frac{c}{(2\pi)^{p/2}} \exp\{\ell^0(\theta) + \varphi s\}|\tilde{\jmath}_{\varphi}|^{-1/2}\, ds \\ &= \frac{c}{(2\pi)^{p/2}} \exp\{\ell^0(\theta) + \varphi s\}|\tilde{\jmath}_{\varphi}|^{1/2}\, d\hat{\varphi} \end{aligned} \tag{3.7}$$

where $\tilde{\jmath}$ is the observed information form the tilted likelihood in the exponent. This is the tangent exponential model (Fraser, 1990) which gives a primary basis for the inference methods in Fraser & Reid (1993b).

4. LOCATION MODEL APPROXIMATION

Again we consider the location-scale standardized model (2.4) and now transform the parameter and variable towards the location type indicated by (1.2).

With a pure location model $\ell(\theta, y) = \ln f(y - \theta)$ we have $\ell_\theta + \ell_y = 0$. We first examine this location property to first derivative at the expansion point $y = 0$. Let $\ell(\theta) = \ell(\theta; 0)$ be the likelihood at the data $y = 0$, $\{\ell^1(\theta), \ldots, \ell^p(\theta)\} = (\partial/\partial y')\ell(\theta; y)\Big|_{y=0}$ be the gradient with respect to y at $y = 0$ and $\{\ell_1(\theta), \ldots, \ell_p(\theta)\} = (\partial/\partial\theta)\ell(\theta; 0)$ be the gradient with respect to θ at $y = 0$. We then define a non-location measure at $y = 0$: $d(\theta) = \{d^1(\theta), \ldots, d^p(\theta)\}$ where

$$\begin{aligned} d^i(\theta) &= \ell^i(\theta) + \ell_i(\theta) \\ &= d^i_{jk}\theta_j\theta_k/2n^{1/2} + d^i_{jk\ell}\theta_j\theta_k\theta_\ell/6n \end{aligned} \tag{4.1}$$

$$d^i_{jk} = a^i_{jk} + a_{ijk}\ , \ d^i_{jk\ell} = a^i_{jk\ell} + a_{ijk\ell} \tag{4.2}$$

and summation is assumed for $ijkl$ over $\{1,\ldots,p\}$. A model then is first derivative location at $y=0$ if $d(\theta)=0$; this expresses the gradient propery above. In addition a model is first derivative location at $y=0$ to order $O(n^{-3/2})$ if and only if $d^i_{jk}=0$, $d^i_{jk\ell}=0$, that is, $a^i_{jk}+a_{ijk}=0$, $a^i_{jk\ell}+a_{ijk\ell}=0$ to order $O(n^{-1/2})$.

We now examine whether a parameter transformation can produce a new model form that satisfies the $y=0$ location property (4.2). For this we first examine terms in (4.1) of order $O(n^{-1/2})$ and consider a transformation

$$\theta_i=\varphi_i+b^i_{jk}\varphi_j\varphi_k/2n^{1/2}\ . \tag{4.3}$$

Substitution in $\ell(\theta)$ and $\ell^i(\theta)$ produces the old nonlocation measure $d^i(\varphi)=d^i_{jk}\varphi_i\varphi_j\varphi_k/2n^{1/2}$ as re-expressed in terms of the new parameter plus an adjustment term; the first coordinate, say, of the adjustment is

$$b^1(\varphi)=\frac{1}{2n^{1/2}}\Big\{-\frac{\partial}{\partial\varphi_1}b^i_{jk}\varphi_i\varphi_j\varphi_k+b^1_{ij}\varphi_i\varphi_j\Big\} \tag{4.4}$$

and is due to the $O(n^{-1/2})$ terms in (4.3). This first coordinate (4.4) can be reexpressed as

$$\frac{1}{2n^{1/2}}\Big\{-2b^1_{11}\varphi_1^2-2(b^1_{1i}+b^i_{11})\varphi_1\varphi_i-(b^i_{1j}+b^j_{1i})\varphi_i\varphi_j\Big\} \tag{4.5}$$

where i, j now run over $\{2,\ldots,p\}$. We now add (4.5) to the first coordinate

$$\frac{1}{2n^{1/2}}\Big\{d^1_{11}\varphi_1^2+2d^1_{1i}\varphi_1\varphi_i+d^1_{ij}\varphi_i\varphi_j\Big\}$$

of (4.1) and a zero nonlocation effect requires the coefficients to be zero; this gives

$$2b^1_{11}=d^1_{11}\ ,\ (b^1_{1i}+b^i_{11})=d^1_{1i}\ ,\ b^i_{1j}+b^j_{1i}=d^1_{ij}$$

The first equation gives $b^1_{11}=\frac{1}{2}d^1_{11}$ and thus in general gives

$$b^\alpha_{\alpha\alpha}=\frac{1}{2}d^\alpha_{\alpha\alpha}. \tag{4.6}$$

The third equation with $i = j$ gives $b^i_{1i} = d^1_{ii}/2$ and thus in general with α, β, assumed not equal, gives

$$b^\beta_{\alpha\beta} = \frac{1}{2} d^\alpha_{\beta\beta} \ . \tag{4.7}$$

The second can be used with (4.7) to give $b^i_{11} = \frac{1}{2}(2d^1_{1i} - d^i_{11})$, and thus

$$b^\beta_{\alpha\alpha} = \frac{1}{2}(2d^\alpha_{\alpha 1} - d^\beta_{\alpha\alpha}) \ . \tag{4.8}$$

The third equation with $i \neq j$ say 2, 3 can then be used with two permutations of its coordinates to give $b^3_{12} + b^2_{13} = d^1_{23}$, $b^1_{23} + b^2_{13} = d^3_{12}$, $b^1_{12} + b^3_{23} = d^2_{13}$ which solve easily to give the general

$$b^\gamma_{\alpha\beta} = \frac{1}{2}(d^\alpha_{\beta\gamma} + d^\beta_{\alpha\gamma} - d^\gamma_{\alpha\beta}) \tag{4.9}$$

With the preceding values for the b's used in (4.3) we suppress the $n^{-1/2}$ terms in the revised nonlocation measure (4.1).

We now assume that the $n^{-1/2}$ terms in (4.1) have been eliminated by the chosen trasformation (4.3), and examine a transformation

$$\theta_i = \varphi_i + b^i_{jk\ell}\varphi_j\varphi_k\varphi_\ell/6n \tag{4.10}$$

with the objective of eliminating the n^{-1} terms in the modified nonlocation measure (4.1). Substitution in $\ell(\theta)$ and $\ell'(\theta)$ produces the old nonlocation measure $d^i(\varphi) = d^i_{jk\ell}\varphi_i\varphi_j\varphi_k\varphi_\ell/n$ as re-expressed in terms of the new parameter plus an adjustment term;, the first coordinate, say, of the adjustment is

$$b^1(\varphi) = \frac{1}{6n}\Big\{ - \frac{\partial}{\partial\varphi_1} b^i_{jk\ell}\varphi_i\varphi_j\varphi_k\varphi_\ell + b^1_{jk\ell}\varphi_j\varphi_k\varphi_\ell\Big\} \tag{4.11}$$

and is due to the $O(n^{-1})$ terms in (4.10). This first coordinate (4.11) can be reexpressed as

$$\begin{aligned} \frac{1}{6n}\{&-3b^1_{11i}\varphi_1^3 - (6b^1_{11i} + 3b^i_{11i})\varphi_1^2\varphi_i \\ &- (3b^1_{1ij} + 3b^i_{11j} + 3b^j_{11i})\varphi_1\varphi_i\varphi_j \\ &+ (b^i_{1jk} + b^j_{1ik} + b^k_{1ij})\varphi_i\varphi_j\varphi_k\} \end{aligned} \tag{4.12}$$

where i, j, k run over $\{2,\ldots,p\}$. We now add (4.12) to the first coordinate

$$\frac{1}{6n}\Big\{d^1_{11i}\varphi_1^3 i + 3d^1_{11i}\varphi_1^2\varphi_i + 3d_{1ij}\varphi_1\varphi_i\varphi_j + d^1_{ijk}\varphi_i\varphi_j\varphi_k\Big\} ,$$

of (4.1), and require a zero non-location effect; this gives

$$\begin{gathered} 3b^1_{11i} = d^1_{11i} \ , \ 2b^1_{11i} + b^i_{11i} = d^1_{11i} \ , \ b_{1ij} + b^i_{11j} + b^j_{11i} = d_{1ij} \\ b^i_{1jk} + b^j_{1ik} + b^k_{1ij} = d^1_{ijk} \end{gathered} \tag{4.13}$$

These equations can be solved as after (4.5) giving

$$\begin{aligned} &b^\alpha_{\alpha\alpha\alpha} = \frac{1}{3}d^\alpha_{\alpha\alpha\alpha} \ , \ b^\alpha_{\beta\beta\beta} = \frac{1}{3}(3d^\beta_{\alpha\beta\beta} - 2d^\alpha_{\beta\beta\beta}) \\ &b^\alpha_{\alpha\beta\beta} = \frac{1}{3}(2d^\beta_{\alpha\alpha\beta} - d^\alpha_{\alpha\beta\beta}) \ , \ b^\alpha_{\alpha\beta\gamma} = \frac{1}{3}(d^\beta_{\alpha\alpha\gamma} + d^\gamma_{\alpha\alpha\beta} - d^\alpha_{\alpha\beta\gamma}) \\ &b^\alpha_{\beta\beta\gamma} = \frac{1}{3}(d^\gamma_{\alpha\beta\beta} - 2d^\alpha_{\beta\beta\gamma} + 2d^\beta_{\alpha\beta\gamma}) \ , \ b^\alpha_{\beta\gamma\delta} = (d^\beta_{\alpha\gamma\delta} + a^\gamma_{\alpha\beta\delta} + d^\delta_{\alpha\beta\gamma} - d^\alpha_{\beta\gamma\delta}) \end{aligned} \tag{4.14}$$

The preceding parameter change places the model (2.4) in the following form

$$\begin{pmatrix} a & a_\alpha & a_{\alpha\beta} & \alpha_{\gamma\beta\gamma} & a_{\alpha\beta\gamma\delta} \\ 0 & I_{i\alpha} & a_{\alpha\beta i} & \alpha_{\alpha\beta\gamma i} & - \\ -I_{ij} & \alpha_{ij\alpha} & a_{\alpha\beta ij} & - & - \\ -\alpha_{ijk} & \alpha_{ijk\alpha} & - & - & - \\ -\alpha_{ijk\ell} & - & - & - & - \end{pmatrix} \tag{4.15}$$

We can now change the variable $x_i = y_i + b^i_{jk}y_jy_k/2n^{1/2} + b^i_{jk\ell}y_jy_ky_\ell/6n$ to make the second row follow the location model form of the first and second column. This can be done in two steps: collapse the variable to have only a linear term as in Section 3; then redefine the variable to obtain the location form. The model then becomes

$$\begin{pmatrix} a & a_\alpha & a_{\alpha\beta} & \alpha_{\gamma\beta\gamma} & a_{\alpha\beta\gamma\delta} \\ 0 & I_{i\alpha} & -\alpha_{i\alpha\beta} & \alpha_{i\alpha\beta\gamma} & - \\ -I_{ij} & \alpha_{ij\alpha} & -a_{ij\alpha\beta} + c_{ij\alpha\beta} & - & - \\ -\alpha_{ijk} & \alpha_{ijk\alpha} & - & - & - \\ -\alpha_{ijk\ell} & - & - & - & - \end{pmatrix} \tag{4.16}$$

If the nonlocation characteristics $c_{ij\alpha\beta}$ are equal to zero, the model is exponential and the coefficients in the first row for the θ_0 density are determined by formulas in Fraser & Reid, (1993b). For non zero coefficients $c_{ij\alpha\beta}$, the normal $(0, I)$ distribution determines the order $O(n^{-1})$ adjustments to the first row of the exponential case.

Thus an asymptotic model with p dimensional variable and p dimensional parameter can be expressed in the canonical location type form with coefficients as in (4.16). It follows that an asymptotic model can be approximated to order $O(n^{-1})$ by a location model on a compact set for the standardized variable and to order $O(n^{-3/2})$ by an location model in a first derivative neighbourhood of the data point, except for a constant of order $O(n^{-1})$.

5. CONSTRUCTION OF CANONICAL PARAMETERS

For third order asymptotic statistical infernece it suffices to have the observed likelihood function and its sample space gradient in an appropriate direction (Fraser & Reid, 1993b). To then apply methods appropriate to exponential or location models it is necessary to obtain the corresponding parametrization for the approximating model.

For the exponential model approximation developed in Section 3, the reparametrization is given explicitly at (3.6) by the sample space gradient of the likelihood function.

For the location approximation developed in Section 4, we have in effect only an existence result given by (4.6)-(4.9) and (4.14). We now discuss briefly the derivation of an invariant formula for the location reparametrization.

As a preliminary step we reparametrize in the pattern (3.6) appropriate to the exponential approximation; with this parametrization, say ϕ, the expansion takes the form (3.4). Now let χ designate the desired location approximation parametrization discussed in Section 4. Then with likelihood re-expressed in terms of the intermediate parameter ϕ, the location relationship for χ obtained from (4.1) has the form

$$\ell_{\phi}(\phi)\partial\phi'/\partial\chi = -\phi \tag{5.1}$$

where parameter vectors are represented as rows and $l_{\phi} = \partial\ell/\partial\phi$.

We now use (5.1) to define χ along an arbitrary ray $\phi = \delta\alpha$ where δ is a scale and α is a unit vector. For an increment $\mathrm{d}\phi$ in ϕ in the direction α, equation (5.1) gives the image increment $\mathrm{d}\chi$ in χ: thus $(\delta\alpha_1, \ldots, \delta\alpha_p)$ has image $\{-\ell_1(\delta\alpha), \ldots, -\ell_p(\delta\alpha)\}$, and the increment $\mathrm{d}\theta = \mathrm{d}\delta \cdot \alpha$ has image $(\mathrm{d}\delta/\delta)\{-\ell_1(\delta\alpha), \ldots, -\ell_p(\delta\alpha)\}$. It follows that

$$\chi = \int_0^\delta -\{\ell_1(s\alpha), \ldots, \ell_p(s\alpha)\} \frac{\mathrm{d}s}{s} \tag{5.2}$$

defines the new parameter along the ray in the direction α.

In the $p = 1$ case we can avoid the intermediate parameterization and obtain the location parameterization χ at $y = 0$

$$\chi = \int_0^\theta -\frac{\ell_\theta(s;0)}{\ell_{;y}(s;0)} \mathrm{d}s. \tag{5.3}$$

These formulas assume the preliminary centering discussed in Section 2.

6. DISCUSSION

An exact or approximate ancillary can reduce a general asymptotic model to a conditional asymptotic model with variable and parameter of the same dimension p; the construction of such ancillaries has been developed in Fraser and Reid (1993b). The reduced model can then be approximated by an exponential model (Section 3) or a location model (Section 4) to obtain $O(n^{-3/2})$ significance formulas for component parameters. The exponential approximation leads to significance formulas of the Lugannani & Rice (1990) type and the location approximation provides the basis for isolating the distribution appropriate for assessing a component parameter.

The parameter transformation that gives location form provides at the same time a likelihood function that gives significance values as areas or volumes under the likelihood function. This has special advantages for presenting information in an easily understood manner and has been strongly recommended by D.A. Sprott.

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