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A two-state jump model

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Abstract

We introduce a pricing model for equity options in which sample paths follow a variance-gamma (VG) jump model whose parameters evolve according to a two-state Markov chain process. As in GARCH type models, jump sizes are positively correlated to volatility. The model is capable of justifying the observed implied volatility skews for options at all maturities. Furthermore, the term structure of implied VG kurtosis is an increasing function of the time to maturity, in agreement with empirical evidence. Explicit pricing formulae, extending the known VG formulae, for European options are derived. In addition, a resummation algorithm, based on the method of lines, which greatly reduces the algorithmic complexity of the pricing formulae, is introduced. This algorithm is also the basis of approximate numerical schemes for American and Bermudan options, for which a state dependent exercise boundary can be computed.

1. Introduction

Several lines of research in pricing theory involve the extension of the pioneering work of Black and Scholes (1973) and Merton (1976). Black–Scholes and Merton had demonstrated that the price of plain vanilla options could be reduced to the estimation of a single quantity—the volatility of the underlying stock, under the assumption that the underlying stock follows a geometric Brownian process. Much effort has been put into improving this basic model with the hope of removing three inherent biases: volatility smiles, skewness and term structure premiums. These biases indicate the market incongruity with the assumption that the underlying stock follows a geometric Brownian process. There are in essence three distinct directions that researchers have followed: one line postulates that the volatility parameter depends on stock prices and attempts to capture the correlation between asset price levels and volatility. Non-parametric specifications have been investigated by Derman and Kani (1994) and Dupire (1994). Closed form generalizations of the Black–Scholes formula are based on the constant-elasticity-of-variance (CEV) model by Cox and Ross (1976) or on the three-parameter family of quadratic volatility models in the works of Ingersoll (1997) and Rady (1997); the two five-parameter families recently found by Albanese *et al* (2001a) are an alternative which include as particular cases, and extend, both quadratic and CEV models. A second line of research attempts to model the volatility as a stochastic process correlated with stock price returns, either through independent risk factors as in the models by Hull and White (1987) or through a GARCH model as in Duan (1996). A third line is based on the postulate that the underlying follows a jump process, as in the variance-gamma (VG) model introduced in Madan *et al* (1998). Jump models are able to explain the skew of implied volatility but fail to capture the term structure of implied kurtosis, a feature that is better explained by stochastic volatility models. State dependent volatility models exploit the correlation between volatility and price movements and are well suited for providing real world usable hedging strategies.

The various families of pricing models are complementary and capture different empirical aspects of stock price processes. Combining the features of the various strands of research under one framework is interesting but presents technical challenges. In this paper we study a model which combines jumps with stochastic volatility, and develop new techniques that drastically reduce the complexity in the calculation of prices to a practical and implementable level.

Konikov and Madan (2000) introduce an extension of the VG model in which the parameters switch, according to a two-state Markov chain, between two fixed sets of values at infinitesimal time intervals. In this paper, we consider a model defined in a similar manner, except that switching occurs at finite time intervals, which in applications have a typical duration of about 3–5 weeks. In this setting, it is still possible to derive pricing formulae in closed form for European options. However, the resulting expressions have a complex combinatorial structure whose numerical evaluation is not practically viable. To overcome this difficulty, we introduce a resummation scheme that greatly reduces the algorithmic complexity by exploiting systematic sign cancellations between the many terms in the combinatorial expression and reduces the computational complexity to an acceptable level. Our algorithm is based on the *model of lines*, introduced by the present authors (Albanese *et al* 2001b, 2001c) to streamline the pricing problem for the ordinary VG model. The model of lines has the additional advantage of allowing approximate solutions for the prices of barrier, American and Bermudan options to be constructed. Since the volatility is locally constant, a double boundary, one for each state of the world, defines the optimal exercise policy. The model contains seven parameters that allow the adjustment of several moments of the return distributions and allows the term structure of implied VG kurtosis to be matched.

2. The VG model

To begin, it is instructive to review some of the salient features of the standard VG model (for further details the reader is referred to Madan *et al* (1998)). The VG process is given by a Brownian motion evaluated at a random time, driven by a gamma process. The stock price process, in the risk-neutral measure, is postulated to be

$$S_t = S_0 \exp\{\omega t + X_{\Gamma_t(\nu)}(\theta;\sigma)\}$$

where $X_\tau(\theta;\sigma)$ is a Brownian process evaluated at time τ with drift θ and volatility σ; $\Gamma_t(\nu)$ is a gamma process evaluated at time t with variance rate ν, mean rate unity and density function given by

$$f_{\Gamma_t(\nu)}(g) = \frac{g^{\frac{t}{\nu}-1}\mathrm{e}^{-\frac{g}{\nu}}}{\Gamma(\frac{t}{\nu})\nu^{\frac{t}{\nu}}}.$$

The drift parameter ω is chosen so that the discounted stock price is a martingale, i.e. so that risk-neutrality, and hence put-call parity, is maintained,

$$\mathbb{E}[\mathrm{e}^{-R_F t} S_t \mid S_0] = S_0 \Rightarrow \omega = R_F + \frac{1}{\nu}\ln\left(1 - \left(\theta + \frac{1}{2}\sigma^2\right)\nu\right). \quad (1)$$

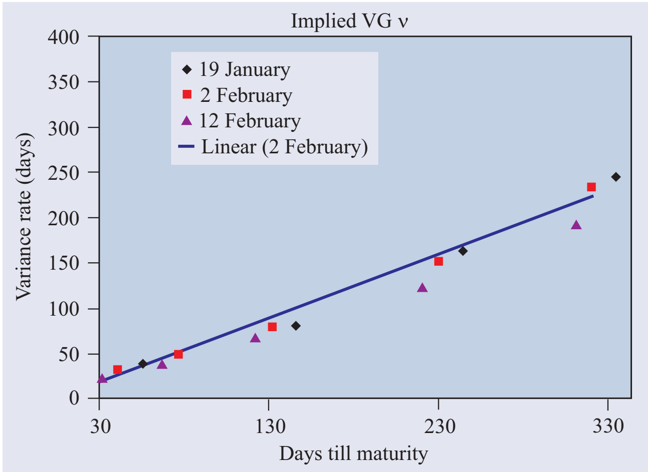

Figure 1. Implied term structure of the variance rate for S&P500 options.

The characteristic function for the VG process can be written as a convolution, by first conditioning on the gamma time change and then integrating over all possible times weighted by the gamma density,

$$\phi_{VG_t(\sigma,\theta,\nu)}(p) = \mathbb{E}[\mathrm{e}^{\mathrm{i}pX_t}] = (1 - \mathrm{i}\theta\nu p + \tfrac{1}{2}\sigma^2\nu p^2)^{-\frac{t}{\nu}}. \quad (2)$$

Although the VG process is built on a diffusion process, it is truly a pure jump process with infinite activity. This can be verified by writing the characteristic function as a product of two gamma process characteristic functions, one with positive jumps and the other with negative jumps.

Just as the limitations of the Black–Scholes model are reflected by the implied volatility skew, the limitations of the VG model are reflected by the term structure of the implied variance rate ν. In figure 1, the term structure of implied ν for options on the S&P500 index is shown. In the following we will demonstrate that this term structure can be explained by the stochastic two-state extension of the VG model proposed in this paper.

3. The two-state variance gamma model

The ability of the VG model to capture implied volatility skews has made it the pricing tool of choice, and it is now widely used in production by leading financial institutions. The model is, however, limited in its ability to replicate the term structure of implied volatility at fixed strike. It tends to predict a decay rate which is substantially higher than the rate implied through quoted option prices. There are many enhancements of the VG model that can help to account for this discrepancy. Adding a stochastic volatility component to the process stands out as an economically meaningful one.

In this paper, we propose a model with two states of the world, each characterized by its own set of VG parameters. In a departure from the model by Konikov and Madan (2000), the time-interval, Δt, over which a single state of the world prevails is postulated to be finite. In applications, Δt would typically be between 3 and 5 weeks. The two possible states are denoted with the subscripts + and −, corresponding to highly volatile and tranquil periods. The volatility, skewness and variance rate in the ± state are taken to be $\sigma_\pm$, $\theta_\pm$ and $\nu_\pm$ respectively. To incorporate an auto-regressive effect into the model, the state probabilities are postulated to be contingent on the current state of the world. Let $q_\pm$ denote the probability that the ± state persists in the next time step. The stock price process over the next period is therefore taken as

$$S_{t+\Delta t|\pm} = S_t \mathrm{e}^{X_{\Delta t|\pm}}$$

contingent on the current state of the world ±. The density functions for $X_{\Delta t|\pm}$ are

$$\begin{aligned} f_{X_{\Delta t}|-}(x) &= q_- f_{VG_{\Delta t}(\sigma_-,\theta_-,\nu_-)}(x - \omega_-\Delta t) \\ &\quad + (1-q_-) f_{VG_{\Delta t}(\sigma_+,\theta_+,\nu_+)}(x - \omega_+\Delta t) \\ f_{X_{\Delta t}|+}(x) &= (1-q_+) f_{VG_{\Delta t}(\sigma_-,\theta_-,\nu_-)}(x - \omega_-\Delta t) \\ &\quad + q_+ f_{VG_{\Delta t}(\sigma_+,\theta_+,\nu_+)}(x - \omega_+\Delta t). \end{aligned} \tag{3}$$

The drifts $\omega_\pm$ appearing in (3) are determined by enforcing risk-neutrality for both states of the world individually and are given by the expression in equation (1) with $\sigma \to \sigma_\pm, \theta \to \theta_\pm$ and $\nu \to \nu_\pm$.

Although the N-step distribution can be written as a convolution, the auto-regressive nature of the process introduces technical complications. Referring the reader interested in the detailed calculations to the appendix, the N-step characteristic function is recorded here,

$$\begin{aligned} \phi_{X_{N\Delta t}|-}(p) &= \sum_{n=0}^{N} \alpha_n^N(q_-, q_+) q_+^{N-n} q_-^n (\mathrm{e}^{\mathrm{i}\omega_-\Delta t p}\phi_-(p))^n \\ &\quad \times (\mathrm{e}^{\mathrm{i}\omega_+\Delta t p}\phi_+(p))^{N-n}. \end{aligned} \tag{4}$$

The state of the world just prior to the next period inception was assumed to be −; the characteristic function when that state is + follows by the replacement $- \leftrightarrow +$ in equation (4). $\phi_\pm(p)$ denotes the conditional VG characteristic function and is given by the expression in equation (2) with $\sigma \to \sigma_\pm$, $\theta \to \theta_\pm$ and $\nu \to \nu_\pm$. The effective combinatorial weights $\alpha_n^N(q_-, q_+)$ are defined as follows:

$$\alpha_n^N(q_-, q_+) = \begin{cases} 1, & n = N \\ ((N-1)Q_+ + 1)Q_-, & n = N-1 \\ \dfrac{1}{Q_+} \displaystyle\sum_{i=2}^{\frac{N+1}{2}-|\frac{N-1}{2}-n|} \binom{N-n-1}{i-1}\binom{n-1}{i-2}(Q_-Q_+)^i & \\ \quad + \left(1 + \dfrac{1}{Q_+}\right) \displaystyle\sum_{i=1}^{\frac{N}{2}-|\frac{N}{2}-n|} \binom{N-n-1}{i-1} & \\ \quad \times \dbinom{n-1}{i-1}(Q_-Q_+)^i + \displaystyle\sum_{i=2}^{\frac{N+1}{2}-|\frac{N+1}{2}-n|} \binom{N-n-1}{i-2} & \\ \quad \times \dbinom{n-1}{i-1}(Q_-Q_+)^{i-1} & 2 \leqslant n \leqslant N-2 \\ ((N-2)Q_-Q_+ + Q_+ + 1)Q_-, & n = 1 \\ Q_-, & n = 0 \end{cases} \tag{5}$$

where the $Q_\pm$ are the probability ratios:

$$Q_- \equiv \frac{(1-q_-)}{q_+} \quad \text{and} \quad Q_+ \equiv \frac{(1-q_+)}{q_-}.$$

Note that when $q_- \to 1 - q_+$, the probability ratios Q_-, $Q_+ \to 1$, and α reduces to the usual binomial combinatorial factor, $\binom{N}{m}$. Consequently, $Q_\pm$ is a measure of auto-regression in the model.

4. The pricing of contingent claims

4.1. European prices—characteristic function method

To ensure the absence of arbitrage, the price of a European style contingent claim is equal to the discounted expectation of the terminal payoff, $\varphi(S_T)$, under the risk-neutral measure $\mathcal{Q}$,

$$P(S_t) = \mathbb{E}^{\mathcal{Q}}[\mathrm{e}^{-R_F(T-t)}\varphi(S_T) \mid S_t; \pm].$$

The N-period characteristic function in (4) can be used to obtain prices of European options within the two-state extension of the VG model. In terms of the characteristic function, the price function at time $N\Delta t$ is

$$\begin{aligned} P(S_t) &= \mathrm{e}^{-R_F N\Delta t} \sum_{n=0}^{N} \alpha_n^N(q_-, q_+) q_+^{N-n} q_-^n \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}y\, \varphi(S\mathrm{e}^y) \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{-\mathrm{i}p(y + x_n^{(N)})} \phi_-^n(p)\phi_+^{N-n}(p) \end{aligned} \tag{6}$$

where

$$x_n^{(N)} \equiv -(n\omega_- + (N-n)\omega_+)\Delta t. \tag{7}$$

The integration over p can be performed as a power series via contour integration when[4] $\Delta t = \nu_+ = \nu_- = \nu$. Such a choice also allows for an elegant solution to path-dependent pricing problems as illustrated in the next section. In that case, denoting the integral over p by $f_n^N(y)$, we find that

$$\frac{(\sigma_-^2\nu_-)^n(\sigma_+^2\nu_+)^{N-n}}{2^N} f_n^N(y) = \begin{cases} -\mathrm{e}^{\delta_-(y+x_n^{(N)})} \displaystyle\sum_{k=0}^{n-1} \frac{(n-1)!}{k!} \mathcal{C}^{n,N-n}_{\delta_-,\gamma_-,\delta_+,\gamma_+}(k)(y + x_n^N)^k & \\ \quad - \mathrm{e}^{\delta_+(y+x_n^{(N)})} \displaystyle\sum_{k=0}^{N-n-1} \frac{(N-n-1)!}{k!} & \\ \quad \times \mathcal{C}^{N-n,n}_{\delta_+,\gamma_+,\delta_-,\gamma_-}(k)(y + x_n^N)^k, & y \geqslant -x_n^{(N)} \\ \mathrm{e}^{\gamma_-(y+x_n^{(N)})} \displaystyle\sum_{k=0}^{n-1} \frac{(n-1)!}{k!} \mathcal{C}^{n,N-n}_{\gamma_-,\delta_-,\gamma_+,\delta_+}(k)(y + x_n^N)^k & \\ \quad + \mathrm{e}^{\gamma_+(y+x_n^{(N)})} \displaystyle\sum_{k=0}^{N-n-1} \frac{(N-n-1)!}{k!}; & \\ \quad \times \mathcal{C}^{N-n,n}_{\gamma_+,\delta_+,\gamma_-,\delta_-}(k)(y + x_n^N)^k, & y < -x_n^{(N)} \end{cases}$$

[4] This can be easily extended to the case when ν_+ is an integer multiple of ν_-

the exponential decay rates are

$$\gamma_\pm = \frac{\theta_\pm + \sqrt{\theta_\pm^2 + 2\frac{\sigma_\pm^2}{\nu}}}{\sigma_\pm^2} > 0$$

and

$$\delta_\pm = \frac{\theta_\pm - \sqrt{\theta_\pm^2 + 2\frac{\sigma_\pm^2}{\nu}}}{\sigma_\pm^2} < 0 \tag{8}$$

and the series coefficients are

$$C^{n_1,n_2}_{\alpha,\beta,\lambda,\kappa}(k) = (-1)^{k+n_2} \sum_{l=0}^{n_1-1-k} \sum_{m=0}^{n_1-1-k-l} \times \frac{(n_1+l-1)!(n_2+m-1)!(n_2+n_1-k-m-l-2)!}{l!m!(n_1-1-k-m-l)!(n_1-1)!((n_2-1)!)^2} \times (\alpha-\beta)^{-(n_1+l)}(\alpha-\lambda)^{-(n_2+m)}(\alpha-\kappa)^{-(n_1+n_2-1-k-m-l)}.$$

Since the density functions $f_n^N(y)$ are polynomials times an exponential weighting factor, the integration over the y-variable in the pricing equation can be carried out explicitly for piece-wise continuous payoffs that are polynomial functions of the terminal stock price. This leads to the final pricing formula.

Although this is a complete closed form solution to the pricing problem, it contains five summations, each of order N, the number of ν time steps to maturity. Furthermore, if the price with maturity $n\Delta t$ is known, this formula does not supply an easy way to compute the price with maturity $(n+1)\Delta t$. Hence, the algorithmic complexity for evaluating the implied volatility surface is $O(N^6)$ polynomial evaluations, where N is the number of ν time steps to maturity. If options of maturity up to 2 years are being valued and ν is of the order of 1 month, the algorithmic complexity is well beyond acceptable limits.

In the next section, we show that a variant of the model of lines *reduces* the algorithmic complexity to $O(N^2)$ polynomial evaluations, rendering calculations practically viable.

4.2. European prices—model of lines method

4.2.1. The model of lines. In Albanese *et al* (2001b, 2001c), the authors developed the *model of lines*, which can be regarded as the analogue of the binomial tree approximation for the VG model. The model of lines provides closed form solutions for VG prices for particular choices of model parameters, *without any approximation error*. In this paper, we extend the model of lines to the two-state stochastic volatility model introduced in the previous section.

The price of a claim in the VG model which matures in time $\Delta t = \nu$, with payoff $\varphi(S_{t+\Delta t})$, is shown in Albanese *et al* (2001b, 2001c) to solve the following ordinary differential equation:

$$-\frac{1}{\Delta t}[P(S_t) - \mathrm{e}^{-R_F\Delta t}\varphi(\mathrm{e}^{\omega\Delta t}S_t)] + D_S P(S_t) = 0$$
$$\text{where} \qquad D_S \equiv \tfrac{1}{2}\sigma^2 S^2\partial_S^2 + (\theta + \tfrac{1}{2}\sigma^2)S\partial_S \tag{9}$$

which when applied recursively computes the price on the nth line and constitutes what was termed the *model of lines*. Note that a discount factor appears in front of the payoff function, ensuring that bonds are correctly priced. The spot price in the payoff function is scaled, so that when conditioning on the financial time the stock drifts with rate θ. Finally, the operator D_S contains no constant term, i.e. the term $-R_F P(S_t)$ is missing in the *model of lines*. On reflection it is clear that such a term must be absent, because in financial time no discounting occurs and risk neutrality holds only in real time. The interested reader is referred to Albanese *et al* (2001b, 2001c) for further details, explanations and a proof of this result.

It is possible to reformulate the model of lines to price instruments in a two-state environment. Consider a European option maturing in a single time step Δt. At the current time, only the present state of the world is known, and the future state has not yet revealed itself. The two conditional prices depending on the current state of the world are (see (3))

$$\boldsymbol{P}_-(S) = \mathrm{e}^{-R_F\Delta t}\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{X_{\Delta t|-}}] = q_- p_-(S) + (1-q_-)p_+(S) \tag{10}$$

$$\boldsymbol{P}_+(S) = \mathrm{e}^{-R_F\Delta t}\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{X_{\Delta t|+}}] = (1-q_+)p_-(S) + q_+p_+(S). \tag{11}$$

Here, the pricing functions $p_\pm(S) \equiv P_{VG}(S, \Delta t, \sigma_\pm, \theta_\pm, \nu_\pm, R_F)$ denote prices conditional on the state of the world over the next time step, and $P_{VG}(S, \tau, \sigma, \theta, \nu, R_F)$ is the price of the claim in the VG model with spot S, time to maturity τ, volatility σ, skewness θ, variance rate ν and risk-free rate R_F.

If the variance rates $\nu_\pm$ are both set equal to the time step Δt, the conditional prices $p_\pm(S)$ must satisfy a differential equation of the form (9),

$$-\frac{1}{\Delta t}[p_\pm(S) - \mathrm{e}^{-R_F\Delta t}\varphi(\mathrm{e}^{\omega_\pm\Delta t}S)] + D_S^\pm p_\pm(S) = 0 \tag{12}$$

where $D_S^\pm$ denotes the differential operator appearing in (9) with $\sigma \to \sigma_\pm$ and $\theta \to \theta_\pm$. These prices can be used to obtain the unconditional prices $\boldsymbol{P}_\pm(S)$. The conditional prices on the next line are then obtained recursively by applying (12) once again with payoff equal to the current unconditional prices: $\varphi(S) = \boldsymbol{P}_\pm(S)$. This leads to the following system of ODEs for the prices on the nth line (the prices are now measured in units of the strike K):

$$\begin{aligned}
D_x^\pm p_\pm^{(n)}(x) &= -\Delta t^{-1}\mathrm{e}^{-R_F\Delta t}\boldsymbol{P}_\pm^{(n-1)}(x+\omega_\pm\Delta t), \\
\boldsymbol{P}_-^{(n)}(x) &= q_- p_-^{(n)}(x) + (1-q_-)p_+^{(n)}(x), \\
\boldsymbol{P}_+^{(n)}(x) &= (1-q_+)p_-^{(n)}(x) + q_+ p_+^{(n)}(x), \\
\boldsymbol{P}_\pm^{(0)}(x) &= \frac{1}{K}\varphi(x)
\end{aligned} \tag{13}$$

where $x \equiv \ln(S/K)$ denotes the moneyness parameter and the differential operators are defined as

$$D_x^\pm \equiv \tfrac{1}{2}\sigma_\pm^2\partial_x^2 + \theta_\pm\partial_x - \Delta t^{-1}. \tag{14}$$

A pictorial representation of this system is shown in figure 2.

4.2.2. The pricing function. The system in equation (13) can be solved explicitly when the payoff is a piecewise continuous polynomial in the terminal stock price. In particular, suppose the payoff is that of a European put $\varphi(x) = (K - \mathrm{e}^x)_+$. Without restricting generality, we can assume that $\omega_+ < \omega_-$. An ansatz which satisfies the differential-difference system as well as the boundary conditions for the conditional prices is

$$
p_-^{(n)} = \begin{cases}
\displaystyle\sum_{m=0}^{n-1}\left\{a_m^{-(n)0}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_1^{(n)})} + c_m^{-(n)0}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_1^{(n)})}\right\}(x-\boldsymbol{x}_1^{(n)})^m, & x > \boldsymbol{x}_1^{(n)} \\
u_1^{-(n)}(\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x) + \displaystyle\sum_{m=0}^{n-1}\Big\{\Big(a_m^{-(n)1}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_2^{n})} + b_m^{-(n)1}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_2^{(n)})}\Big) + \Big(c_m^{-(n)1}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_2^{(n)})} + d_m^{-(n)1}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_2^{(n)})}\Big)\Big\}(x-\boldsymbol{x}_2^{(n)})^m, & \boldsymbol{x}_2^{(n)} < x \leqslant \boldsymbol{x}_1^{(n)} \\
\vdots & \\
u_{n-1}^{-(n)}(\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x) + \displaystyle\sum_{m=0}^{n-1}\Big\{\Big(a_m^{-(n)n-1}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_n^{(n)})} + b_m^{-(n)n-1}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_n^{(n)})}\Big) + \Big(c_m^{-(n)n-1}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_n^{(n)})} + d_m^{-(n)n-1}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_n^{(n)})}\Big)\Big\}(x-\boldsymbol{x}_n^{(n)})^m, & \boldsymbol{x}_n^{(n)} < x \leqslant \boldsymbol{x}_{n-1}^{(n)} \\
\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x + \displaystyle\sum_{m=0}^{n-1}\left\{b_m^{-(n)n}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_{n+1}^{(n)})} + d_m^{-(n)n}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_{n+1}^{(n)})}\right\}(x-\boldsymbol{x}_{n+1}^{(n)})^m, & x \leqslant \boldsymbol{x}_n^{(n)}
\end{cases}
\tag{15}
$$

and

$$
p_+^{(n)} = \begin{cases}
\displaystyle\sum_{m=0}^{n-1}\left\{a_m^{+(n)0}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_0^{(n)})} + c_m^{+(n)0}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_0^{(n)})}\right\}(x-\boldsymbol{x}_0^{(n)})^m, & x > \boldsymbol{x}_0^{(n)} \\
u_1^{+(n)}(\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x) + \displaystyle\sum_{m=0}^{n-1}\Big\{\Big(a_m^{+(n)1}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_1^{(n)})} + b_m^{+(n)1}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_1^{(n)})}\Big) + \Big(c_m^{+(n)1}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_1^{(n)})} + d_m^{+(n)1}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_1^{(n)})}\Big)\Big\}(x-\boldsymbol{x}_1^{(n)})^m, & \boldsymbol{x}_1^{(n)} < x \leqslant \boldsymbol{x}_0^{(n)} \\
\vdots & \\
u_{n-1}^{+(n)}(\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x) + \displaystyle\sum_{m=0}^{n-1}\Big\{\Big(a_m^{+(n)n-1}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_{n-1}^{(n)})} + b_m^{+(n)n-1}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_{n-1}^{(n)})}\Big) + \Big(c_m^{+(n)n-1}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_{n-1}^{(n)})} + d_m^{+(n)n-1}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_{n-1}^{(n)})}\Big)\Big\}(x-\boldsymbol{x}_{n-1}^{(n)})^m, & \boldsymbol{x}_{n-1}^{(n)} < x \leqslant \boldsymbol{x}_{n-2}^{(n)} \\
\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x + \displaystyle\sum_{m=0}^{n-1}\left\{b_m^{+(n)n}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_n^{(n)})} + d_m^{+(n)n}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_n^{(n)})}\right\}(x-\boldsymbol{x}_n^{(n)})^m, & x \leqslant \boldsymbol{x}_{n-1}^{(n)}
\end{cases}
\tag{16}
$$

where the region boundaries, $\boldsymbol{x}_l^{(n)}$, are given by (7). The factors appearing in the exponentials, $-\gamma_\pm$ and $-\delta_\pm$, are the positive and negative roots of the characteristic polynomials, $\mathrm{ch}_\pm(s) = \frac{1}{2}\sigma_\pm^2 s^2 + \theta_\pm s - \Delta t^{-1}$, of the differential operators $D_x^\pm$, and have already appeared in equation (8). There are three main regions in the solution corresponding to the option being in the money with respect to both states of the world (purple region in figure 3), in the money with respect to one and out of the money with respect to the other state of the world (blue region in figure 3) and finally out of the money with respect to both states of the world (red region in figure 3). Furthermore, the prices p_- and p_+ have different upper and lower region boundaries, while the interior ones overlap. There are in general three typical kinds of behaviour for the region boundaries themselves: if $\omega_\pm > 0$ then both boundaries constantly decrease, if $\omega_+ < 0$ while $\omega_- > 0$ then the upper boundary increases while the lower one decreases and if $\omega_\pm < 0$ then both boundaries continually increase. In principle any of the three kinds of behaviour is possible, and as such no assumptions on the signs of $\omega_\pm$ are made. Figure 3 depicts the situation with $\omega_+ < 0$ and $\omega_- > 0$. The intermediate region in this figure is reminiscent of the recombining binomial tree in the CRR model (Cox *et al* 1979).

The corresponding ansätze for the unconditional prices, $\boldsymbol{P}_\pm^{(n)}$, are as follows:

$$
\boldsymbol{P}_\pm^{(n)} = \begin{cases}
\displaystyle\sum_{m=0}^{n-1}\left\{\boldsymbol{a}_m^{\pm(n)0}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_0^{(n)})} + \boldsymbol{c}_m^{\pm(n)0}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_0^{(n)})}\right\}(x-\boldsymbol{x}_0^{(n)})^m, & x > \boldsymbol{x}_0^{(n)} \\
\boldsymbol{u}_1^{\pm(n)}(\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x) + \displaystyle\sum_{m=0}^{n-1}\left\{\left(\boldsymbol{a}_m^{\pm(n)1}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_1^{(n)})} + \boldsymbol{b}_m^{\pm(n)1}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_1^{(n)})}\right) + \left(\boldsymbol{c}_m^{\pm(n)1}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_1^{(n)})} + \boldsymbol{d}_m^{\pm(n)1}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_1^{(n)})}\right)\right\}(x-\boldsymbol{x}_1^{(n)})^m, & \boldsymbol{x}_1^{(n)} < x \leqslant \boldsymbol{x}_0^{(n)} \\
\vdots & \\
\boldsymbol{u}_n^{\pm(n)}(\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x) + \displaystyle\sum_{m=0}^{n-1}\left\{\left(\boldsymbol{a}_m^{\pm(n)n}\mathrm{e}^{-\gamma_+(x-\boldsymbol{x}_n^{(n)})} + \boldsymbol{b}_m^{\pm(n)n}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_n^{(n)})}\right) + \left(\boldsymbol{c}_m^{\pm(n)n}\mathrm{e}^{-\gamma_-(x-\boldsymbol{x}_n^{(n)})} + \boldsymbol{d}_m^{\pm(n)n}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_n^{(n)})}\right)\right\}(x-\boldsymbol{x}_n^{(n)})^m, & \boldsymbol{x}_n^{(n)} < x \leqslant \boldsymbol{x}_{n-1}^{(n)} \\
\mathrm{e}^{-R_F n\Delta t} - \mathrm{e}^x + \displaystyle\sum_{m=0}^{n-1}\left\{\boldsymbol{b}_m^{\pm(n)n+1}\mathrm{e}^{-\delta_+(x-\boldsymbol{x}_{n+1}^{(n)})} + \boldsymbol{d}_m^{\pm(n)n+1}\mathrm{e}^{-\delta_-(x-\boldsymbol{x}_{n+1}^{(n)})}\right\}(x-\boldsymbol{x}_{n+1}^{(n)})^m, & x \leqslant \boldsymbol{x}_n^{(n)}.
\end{cases}
\tag{17}
$$

Inserting the ansätze (15)–(17) into the differential-difference system (13) forces the coefficients for the conditional prices to satisfy the following set of recurrence

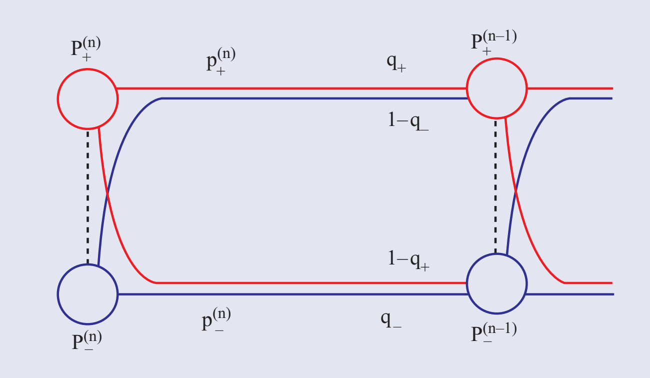

Figure 2. A pictorial representation of the chain of pricing functions in the two-state model of lines.

relations:

$$p_- \begin{cases} a_m^{-(n)i} = \psi_m^-(a^{-(n)i}, \boldsymbol{a}^{-(n)i}, \gamma_+) \\ b_m^{-(n)i} = \psi_m^-(b^{-(n)i}, \boldsymbol{b}^{-(n)i}, \delta_+) \\ c_m^{-(n)i} = \Psi_m^-(c^{-(n)i}, \boldsymbol{c}^{-(n)i}, \gamma_-) \\ d_m^{-(n)i} = \Psi_m^-(d^{-(n)i}, \boldsymbol{d}^{-(n)i}, \delta_-) \\ u_m^{-(n)i} = \boldsymbol{u}_m^{-(n-1)i} \end{cases} \tag{18}$$

$$p_+ \begin{cases} a_m^{+(n)i} = \Psi_m^+(a^{+(n)i}, \boldsymbol{a}^{+(n)i}, \gamma_+) \\ b_m^{+(n)i} = \Psi_m^+(b^{+(n)i}, \boldsymbol{b}^{+(n)i}, \delta_+) \\ c_m^{+(n)i} = \psi_m^+(c^{+(n)i}, \boldsymbol{c}^{+(n)i}, \gamma_-) \\ d_m^{+(n)i} = \psi_m^+(d^{+(n)i}, \boldsymbol{d}^{+(n)i}, \delta_-) \\ u_m^{+(n)i} = \boldsymbol{u}_m^{+(n-1)i} \end{cases} \tag{19}$$

where the functions ψ and Ψ are

$$\psi_m^\pm(z^{(n)}, \boldsymbol{z}^{(n)}, \kappa) = -\frac{\frac{e^{-R_F\Delta t}}{\Delta t}\boldsymbol{z}_{m-1}^{(n-1)} + \frac{1}{2}m(m+1)\sigma_\pm^2 z_{m+1}^{(n)}}{(\theta_\pm - \sigma_\pm^2\kappa)m},$$

$$1 \leqslant m \leqslant n-1$$

$$\Psi_m^\pm(z^{(n)}, \boldsymbol{z}^{(n)}, \kappa) = \begin{cases} -\dfrac{1}{\mathrm{ch}_\pm(-\kappa)}\Big[(\theta_\pm - \sigma_\pm^2\kappa)(m+1)z_{m+1}^{(n)} \\ \quad + \dfrac{1}{2}(m+1)(m+2)\sigma_\pm^2 z_{m+2}^{(n)} + \dfrac{e^{-R_F\Delta t}}{\Delta t}\boldsymbol{z}_m^{(n-1)}\Big], \\ \quad 0 \leqslant m \leqslant n-2 \\ 0, \qquad m = n-1. \end{cases}$$

The coefficients of the unconditional prices are obtained via relations of the form

$$\boldsymbol{a}_m^{-(n)i} = \begin{cases} (1-q_-)a_m^{+(n)0} + q_- e^{-\gamma_+(\boldsymbol{x}_0^{(n)} - \boldsymbol{x}_1^{(n)})} \displaystyle\sum_{l=m}^{n-1} \binom{l}{m} a_l^{-(n)0} \\ \quad \times (\boldsymbol{x}_0^{(n)} - \boldsymbol{x}_1^{(n)})^{l-m}, \qquad i = 0 \\ (1-q_-)a_m^{+(n)i} + q_- a_m^{-(n)i-1}, \qquad 1 \leqslant i \leqslant n \end{cases} \tag{20}$$

for $\boldsymbol{a}^\pm$ and $\boldsymbol{c}^\pm$, while $\boldsymbol{b}^\pm$ and $\boldsymbol{d}^\pm$ are determined by relations of the form

$$\boldsymbol{b}_m^{-(n)i} = \begin{cases} q_- b_m^{-(n)i-1} + (1-q_-)b_m^{+(n)i}, \qquad 1 \leqslant i \leqslant n \\ q_- b_m^{-(n)n} + (1-q_-)e^{-\gamma_+(\boldsymbol{x}_{n+1}^{(n)} - \boldsymbol{x}_n^{(n)})} \displaystyle\sum_{l=m}^{n-1} \binom{l}{m} b_l^{+(n)n} \\ \quad \times (\boldsymbol{x}_{n+1}^{(n)} - \boldsymbol{x}_n^{(n)})^{l-m}, \qquad i = n+1 \end{cases} \tag{21}$$

and finally

$$\boldsymbol{u}_m^{-(n)i} = \begin{cases} q_- u_1^{-(n)} + (1-q_-)u_1^{+(n)}, & i = 1 \\ q_- u_{i-1}^{-(n)} + (1-q_-)u_i^{+(n)}, & 2 \leqslant i \leqslant n-1 \\ q_- u_{n-1}^{-(n)} + (1-q_-)u_{n-1}^{+(n)}, & i = n \end{cases} \tag{22}$$

These recurrence relations are not sufficient to determine the $m = 0$ coefficients obtained by using the function ψ. Rather, those coefficients are determined by enforcing continuity in the conditional pricing functions and their deltas at the boundaries between solution regions. This leads to a simple linear system which can be easily solved.

4.3. American option prices

American style options are priced by solving an optimal stopping time problem. For example, the price of an American put option is

$$\boldsymbol{P}(S, T) = \sup_{\tau \in (0,T]} \mathbb{E}[e^{-R_F\tau}(K - S_\tau)_+ \mid S_0 = S]. \tag{23}$$

The optimal exercise policy is usually described by a monotonically decreasing boundary, $S_B(t)$. When the stock level crosses this boundary, the option is exercised by a rational investor. However, in the two-state extension of the VG model two boundaries, $S_{B^\pm}(t)$, exist. If the world is in the + state, then it is optimal to exercise at the barrier $S_{B^+}(t)$; while, if the world is in the − state, it is optimal to exercise at the barrier $S_{B^-}(t)$. In the + state of the world there is a greater probability of a downward price movement before maturity; consequently, there is a higher probability that the option will be in the money. The boundaries are therefore relatively ordered so that $S_{B^+}(t) \leqslant S_{B^-}(t)$ for all times t.

The solution of the optimal stopping problem (23) is difficult in general, and an exact answer is not available. In Albanese *et al* (2001c) we demonstrated that the model of lines leads to an exact closed form solution when the optimal exercise boundary is assumed to be constant between time steps. However, with two volatility levels, the number of regions quickly becomes unmanageable, and renders analytic techniques computationally useless. Fortunately, a simple numerical procedure combined with the analytic results for the European put resolves this problem.

Just as in the case of European options, the sign of $\omega_\pm$, which is predominantly determined by the skewness parameters $\theta_\pm$, dictates different behaviour of the region boundaries for the conditional American pricing functions. The two main types of behaviour are (a) $\omega_+ < 0$, in which case the upper boundary region is monotonically increasing, and (b) $\omega_+ > 0$, in which case the upper boundary region

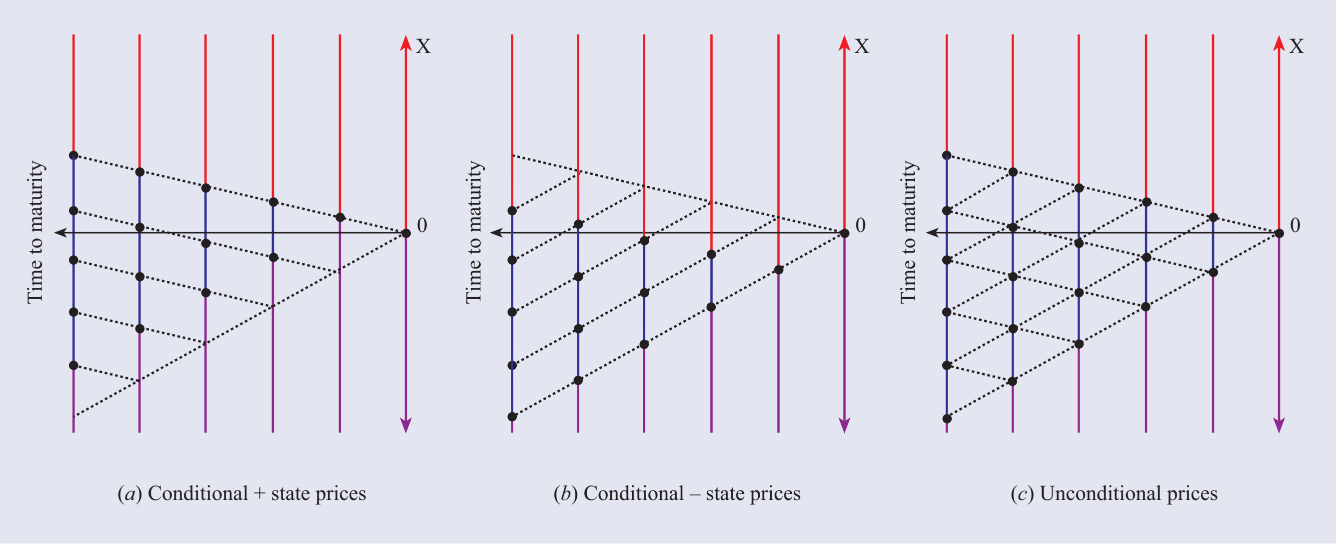

Figure 3. The conditional prices for a European put option are split into $n+1$ regions on the nth line, while the unconditional prices contain $n+2$ regions. This depicts the situation when $\omega_+ < 0 < \omega_-$.

is monotonically decreasing. In case (a) the upper boundary will never cross the American exercise boundaries, while in case (b) the exercise boundaries may lie below the upper boundary at first, but then cross it and eventually remain above the upper boundary. These scenarios are depicted in figure 4. The strategy we follow is to use numerical solutions within the region bounded by the optimal exercise boundary and the upper boundary region (denoted by the blue lines with arrows in figure 4), while using analytic solutions in the two outer regions (denoted by the red and purple regions in 4). Once the American boundary lies above the upper region, the solution becomes entirely analytic. This general idea is now described in some detail.

The methodology is to solve the system (13) with the following boundary conditions:

$$\lim_{S\to\infty} p_\pm^{(n)}(S) = 0,$$

$$\lim_{S\to S_{B\pm}^{(n)}} p_\pm^{(n)}(S) = K - S_{B\pm}^{(n)},$$

$$\lim_{S\to S_{B\pm}^{(n)}} d_S p_\pm^{(n)}(S) = -1.$$

At each step the two boundaries $S_{B\pm}^{(n)}$ must be solved for, and once they are obtained the price at the period inception, before the state of the world reveals itself, is given by the unconditional price as in (10) and (11).

By numerically integrating the ODE which determines the conditional prices on the nth line in terms of the price on the $(n-1)$th line, equation (13), from the optimal exercise boundaries, $S_{B\pm}^{(n)}$, where the prices are equal to the intrinsic value of the option and the deltas must be equal to minus one, up to the uppermost boundary region, $x_0^{(n)}$ for p_+ and $x_1^{(n)}$ for p_-, the conditional prices and deltas at those points will be known. Denote the numerically obtained prices and deltas at the two upper boundaries by $p_{0\pm}^{(n)}$ and $\Delta_{0\pm}^{(n)}$ respectively. Since the in-the-money price of an American style option is equal to the intrinsic value of the option, $K-S$, only the out-of-the-money price remains to be calculated. The price functions have the same form as the out-of-the-money European prices found in equations (15)–(17), and the coefficients must still satisfy the recurrence relations (18)–(21). The undetermined $m=0$ coefficients can be obtained in terms of the numerically integrated results $p_{0\pm}^{(n)}$ by forcing the continuity conditions on the prices,

$$a_0^{-(n)0} = p_{0-}^{(n)} - c_0^{-(n)0} \qquad \text{and} \qquad c_0^{+(n)0} = p_{0+}^{(n)} - a_0^{+(n)0}.$$

The recurrence relations together with the above two equations fully determine the out-of-the-money price of the American option up to the numerical constants $p_{0\pm}^{(n)}$, which, in addition to the deltas, are fixed in terms of the numerical integration from the exercise boundary. Since the conditional deltas must also be continuous at both upper boundaries, the deltas must also satisfy the following equations:

$$\Delta_{0-}^{(n)} = \gamma_+ a_0^{-(n)0} - a_1^{-(n)0} + \gamma_- c_0^{-(n)0} - c_1^{-(n)0}$$
$$\Delta_{0+}^{(n)} = \gamma_+ a_0^{+(n)0} - a_1^{+(n)0} + \gamma_- c_0^{+(n)0} - c_1^{+(n)0}.$$

For arbitrary choices of the exercise boundaries, $S_{B\pm}^{(n)}$, these two equations will not be satisfied. The determination of the optimal exercise boundaries can therefore be made by varying them until the above continuity equations are satisfied.

We find that this combination of analytic results and a numerical integration to solve for the exercise boundary leads to an efficient and simple to implement algorithm. In figure 5 the optimal exercise boundaries for an option on a stock that follows our two-state volatility model are illustrated.

The pricing of barrier options can be carried out much along the same lines as the American case, except that now the exercise boundaries are fixed and equal in both states of the world. Analytical results for barriers are available; however, the numerical/analytical scheme proves to be more versatile.

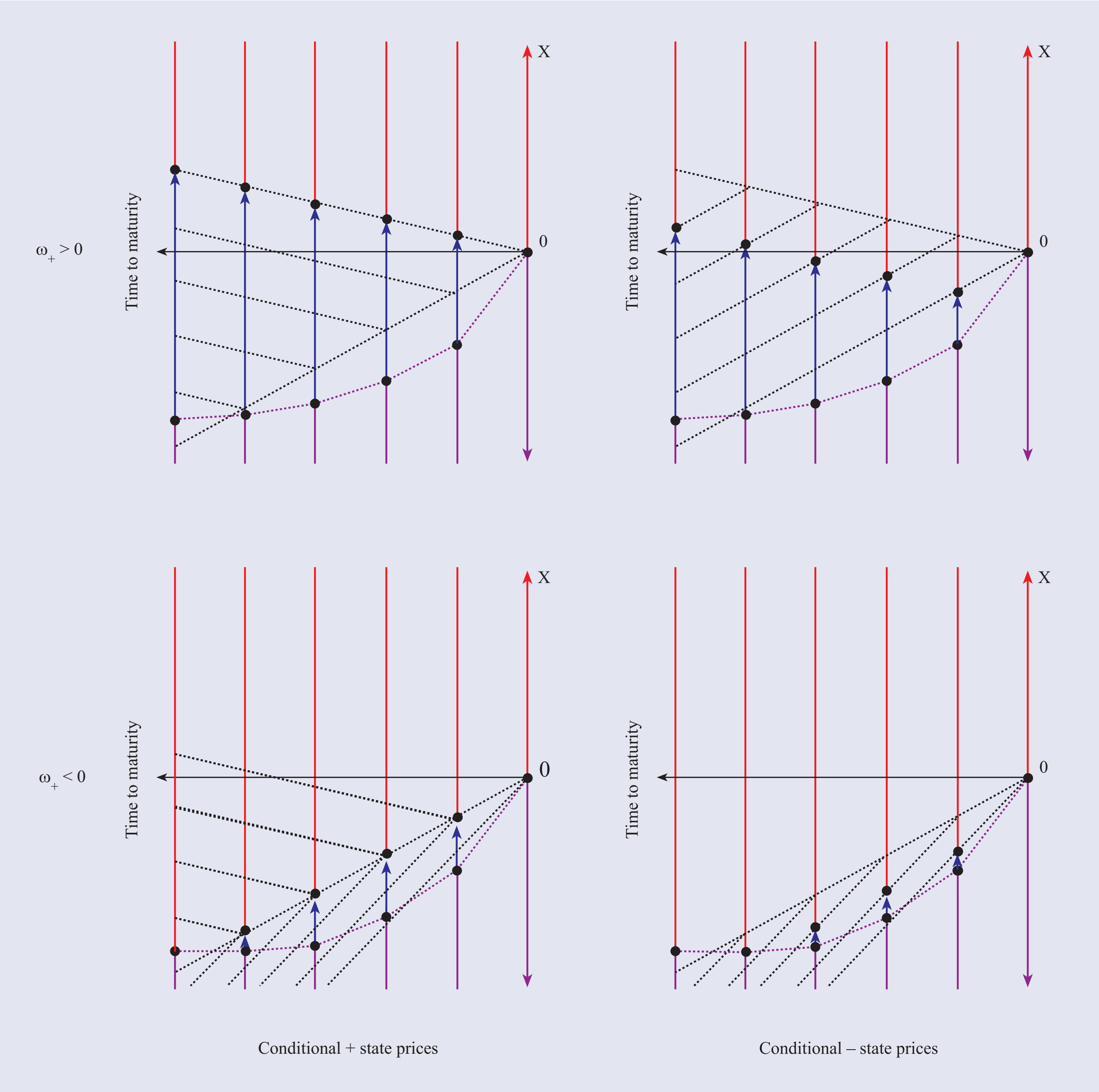

Figure 4. The multiple regions that appear in the European case can be integrated over numerically to avoid complications for American options.

5. Numerical results and conclusions

We find that the two-state extended VG model discussed in this paper is able to explain both implied volatility skews and the observed term structure for implied variance rate. This is illustrated in figures 6 and 7, where the Black–Scholes implied volatilities of the prices calculated within our model are plotted against the strike K and the ratio $\log(F/K)/\sqrt{T}$ respectively, where F is the forward price of the underlying and T is the option maturity. In this example, the spot price is \$100, the continuously compounded interest rate is 5%, the number of lines is 16, the number of days between lines and the ν parameter is 28 days, the volatility can take the values 10% and 30% and the VG skewness parameter is set to 0 in the plus state and -20% in the minus state.

The term structure of implied VG variance rate is given in figure 8. Notice that the upward slope and almost linear behaviour over a period of 16 months reflects the observed features of market prices, as captured by the graph in figure 1.

When calibrating our model to market prices, one has to address the issue of marking to market the spot value of volatility. A reasonable way of accomplishing this task is to adjust the probability amplitudes of the first period separately from the ones in the following period, which can all be

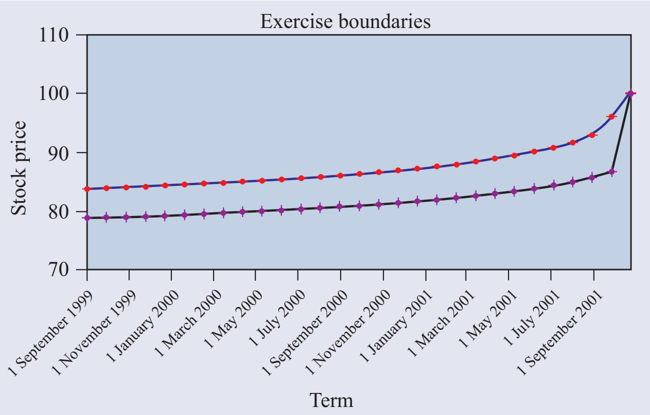

Figure 5. Double exercise boundary corresponding to volatilities 30% and 10% and biweekly switch rate.

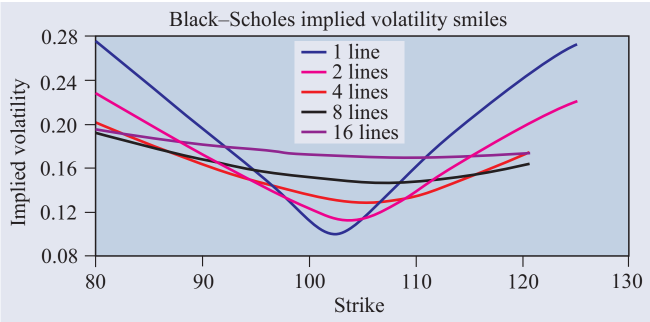

Figure 6. Implied Black–Scholes volatility of theoretical two-state VG prices versus the strike price.

postulated to be equal. The criterion for the adjustment would be to match the price of a short-dated at-the-money option. The interpretation of this calibration procedure would be to express a view about the current level of volatility by assigning the probability q that the $-$ state will be realized and the probability $1-q$ that the $+$ state will occur. In our formalism this amounts to choosing $q_+ = 1-q_- = 1-q$ for *the first period only*. After that, the volatility process retains its auto-regressive nature with stationary transition probabilities.

Finally, the two American exercise boundaries in our model are given in figure 5. The parameters in this example are identical to those in figures 6 and 7.

In conclusion, in this paper we study the technical aspects of a pricing model for equity options in which sample paths follow a variance-gamma (VG) jump model whose parameters evolve according to a two-state Markov chain. The main result of the paper is a resummation algorithm based on the method of lines, which greatly reduces the algorithmic complexity of the pricing formulae. This algorithm is also the basis of approximate numerical schemes for American and Bermudan options, for which a state-dependent exercise boundary can be computed. We also show that the model is capable of justifying the observed implied volatility skews for options at all maturities and the term structure of implied variance rate appears to be an increasing function of the time to maturity, in agreement with empirical evidence.

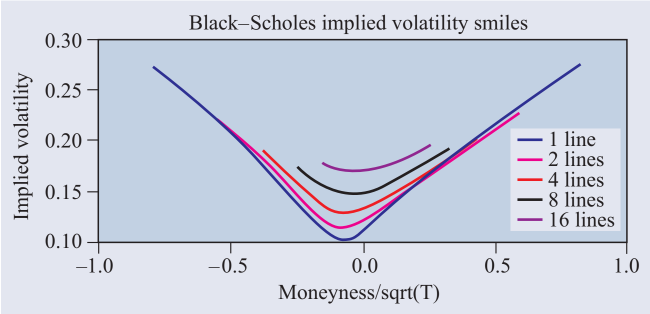

Figure 7. Implied Black–Scholes volatility of two-state VG prices versus the ratio $\log(F/K)/\sqrt{T}$.

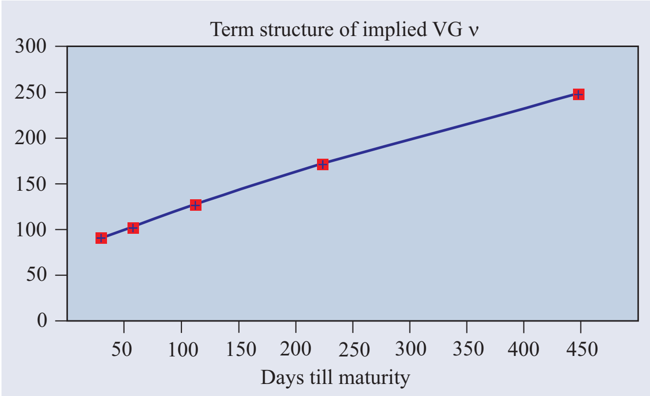

Figure 8. Term structure of the implied VG variance-rate as a function of the option's maturity.

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Appendix

This appendix contains the derivation of the N-step characteristic function given in equation (4).

Assume that the current state of the world is $-$, then the paths of the stochastic variables can be characterized in four ways:

(a) over the next step the world is in the $+$ state and the world ends in the $+$ state;
(b) over the next step the world is in the $+$ state and the world ends in the $-$ state;
(c) over the next step the world is in the $-$ state and the world ends in the $-$ state and
(d) over the next step the world is in the $-$ state and the world ends in the $+$ state.

Enumerate the states of the world by collecting how many $+$ states are in a row and how many $-$ states, and so on. For example, suppose the state of world takes the following path:

$$- \mid +++++---++----+-++$$

where the vertical line indicates the current time. This path falls within category 1, and it is enumerated as

$$- \mid [5][3][2][4][1][1][2].$$

The numbers in the brackets indicate the number of + states and − states alternately. In general, the states, from category 1, can therefore be enumerated as follows:

$$- \mid [p_1][m_1][p_2][m_2][p_3][m_3][p_4][m_4]\ldots[p_{i-1}][m_{i-1}][p_i];$$

the probability weight for this configuration will therefore be

$$P_1 = (1-q_-)[(1-q_-)^{i-1}q_-^{m_1+\cdots+m_{i-1}-i+1}(1-q_+)^{i-1}q_+^{p_1+\cdots+p_i-i}].$$

Of course there are constraints on i, $\{p_1,\ldots,p_i\}$ and $\{m_1,\ldots,m_{i-1}\}$:

$$\begin{aligned}
&p_j \geqslant 1, \qquad j = 1,\ldots,i\\
&m_j \geqslant 1, \qquad j = 1,\ldots,i-1\\
&(p_1+\cdots+p_i)+(m_1+\cdots+m_{i-1}) = N\\
&\quad 1 \leqslant i \leqslant \frac{N+1}{2}.
\end{aligned}$$

Since only the sums $p = p_1+\cdots+p_i$ and $m = m_1+\cdots+m_{i+1}$ appear in the probability and constraints, it is convenient to rewrite the measure in terms of these variables. However, all i partitions of p and $(i-1)$ partitions of m must be accounted for. It is easy to show that the multiplicity factors are $\binom{p-1}{i-1}$ and $\binom{m-1}{i-2}$ respectively. Consequently, all category 1 paths have the following characteristic function:

$$\begin{aligned}
\phi_1(p) &= (1-q_-)q_+^{N-1}(\mathrm{e}^{\mathrm{i}\omega_+\Delta t p}\phi_+(p))^N\\
&+ \sum_{i=2}^{(N+1)/2}\Bigg\{(1-q_-)^i(1-q_+)^{i-1}\\
&\times \sum_{m=i-1}^{N-i}\binom{N-m-1}{i-1}\binom{m-1}{i-2}\\
&\times q_+^{N-m-i}q_-^{m-i+1}(\mathrm{e}^{\mathrm{i}\omega_-\Delta t p}\phi_-(p))^m(\mathrm{e}^{\mathrm{i}\omega_+\Delta t p}\phi_+(p))^{N-m}\Bigg\}.
\end{aligned}$$

The first term in this expression is responsible for the boundary case in which the world switches into state + and never switches after that (i.e. $i = 1$).

An enumeration of category 2, 3 and 4 paths is similarly given by the following:

$$\begin{aligned}
&- \mid [p_1][m_1][p_2][m_2]\cdots[p_i][m_i]\\
&- \mid [m_1][p_1][m_2][p_2]\cdots[m_{i-1}][p_{i-1}][m_i]\\
&- \mid [m_1][p_1][m_2][p_2]\cdots[m_i][p_i].
\end{aligned}$$

These paths occur with probabilities

$$\begin{aligned}
P_2 &= (1-q_-)[(1-q_-)^{i-1}q_-^{m_1+\cdots+m_i-i}(1-q_+)^i q_+^{p_1+\cdots+p_i-i}]\\
P_3 &= q_-[(1-q_-)^{i-1}q_-^{m_1+\cdots+m_i-i}(1-q_+)^{i-1}q_+^{p_1+\cdots+p_{i-1}-i+1}]\\
P_4 &= q_-[(1-q_-)^i q_-^{m_1+\cdots+m_i-i}(1-q_+)^{i-1}q_+^{p_1+\cdots+p_i-i}]
\end{aligned}$$

respectively. The constraints are analogous to the category 1 case, with obvious adjustments. Their contributions to the probability density follow straightforwardly,

$$\begin{aligned}
\phi_2(x) &= \sum_{i=1}^{N/2}\Bigg\{(1-q_-)^i(1-q_+)^i\\
&\times \sum_{m=i}^{N-i}\binom{N-m-1}{i-1}\binom{m-1}{i-1}\\
&\times q_+^{N-m-i}q_-^{m-i}(\mathrm{e}^{\mathrm{i}\omega_-\Delta t p}\phi_-(p))^m(\mathrm{e}^{\mathrm{i}\omega_+\Delta t p}\phi_+(p))^{N-m}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\phi_3(x) &= q_-^N(\mathrm{e}^{\mathrm{i}\omega_-\Delta t p}\phi_-(p))^N\\
&+ \sum_{i=2}^{(N+1)/2}\Bigg\{(1-q_-)^{i-1}(1-q_+)^{i-1}\\
&\times \sum_{m=i}^{N-i+1}\binom{N-m-1}{i-2}\binom{m-1}{i-1}q_+^{N-m-i+1}\\
&\times q_-^{m-i+1}(\mathrm{e}^{\mathrm{i}\omega_-\Delta t p}\phi_-(p))^m(\mathrm{e}^{\mathrm{i}\omega_+\Delta t p}\phi_+(p))^{N-m}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\phi_4(x) &= \sum_{i=1}^{N/2}\Bigg\{(1-q_-)^i(1-q_+)^{i-1}\\
&\times \sum_{m=i}^{N-i}\binom{N-m-1}{i-1}\binom{m-1}{i-1}\\
&\times q_+^{N-m-i}q_-^{m-i+1}(\mathrm{e}^{\mathrm{i}\omega_-\Delta t p}\phi_-(p))^m(\mathrm{e}^{\mathrm{i}\omega_+\Delta t p}\phi_+(p))^{N-m}\Bigg\}.
\end{aligned}$$

It is also possible to reorder the summation over the number of partitions, i, and the number of encountered states of the world. Such a reordering leads to the form of the characteristic function given in equation (4).

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