Recall that \( \mathbb{E}[e^{uZ}] = e^{\frac{1}{2}u^2} \) where \( Z \sim \mathcal{N}(0, 1) \) is a standard normal random variable.
1. Write down concise (about 50 words) but precise responses to each of the following:

(a) [5] What is the fundamental theorem of asset pricing?

\[ \exists \alpha \text{ s.t. } C_0 = e^{-\alpha T} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\gamma T} C_T \right] \]

\[ \forall \text{ traded } C \]

\[ \Leftrightarrow \exists \text{ arbi.} \]

(b) [5] What is unacceptable about the Ho-Lee model of interest rates?

It is normally distributed and its variance grows with T. 

\[ \Rightarrow \text{ high prob of } T < 0. \]
2. [10] Please indicate true or false. no explanations required
   -1 for incorrect answer, +2 for correct answer, 0 for blank answer.

(a) [T] [F]
   The price of a call option always decreases with increasing volatility.

(b) [T] [F]
   If an interest rate model matches bond prices at all maturities, then the risk-neutral branching
   probabilities must equal $\frac{1}{2}$.

(c) [T] [F]
   Suppose a risk-neutral measure exists, then the price of traded assets are unique.

(d) [T] [F]
   The limiting distribution of a discrete time asset price model must always be log-normal.

(e) [T] [F]
   A forward start call option price is linear in the current spot price.
3. Sketch each of the following:

(a) [5] The optimal exercise curves for an American put option with maturity $T = 1$ and strike $K = 1$ for three levels of volatility (i) low (ii) medium and (iii) high.

[sketch the three curves on the same graph, clearly label them and any interesting points.]
(b) [5] The price of a portfolio consisting of 1 long put struck at $1, and 2 short calls struck at
$1 for three maturities: (i) at maturity $T = 0$ (ii) maturity $T = 1$ and (iii) maturity $T = 2$.

[sketch the three curves on the same graph, clearly label them and any interesting points.]
4. (a) You are given that the following 5 paths are the only possible paths of an asset price and that their risk-neutral probabilities are all $\frac{2}{5}$ and $r = 0$.

Write down values (don’t carry out the arithmetic) for each of the following 1-year options

i. /1/ a European call struck at 100.

\[
\frac{5 + 38}{5} \]

ii. /1/ a European Binary put struck at 110.

\[
\frac{4}{5} \]
iii. \([I]\) a down-and-in knock-in call with lower barrier of 80 and a strike of 70.

\[
\frac{14}{5}
\]

iv. \([I]\) an up-and-out knock-out put with upper barrier of 110 and a strike of 100.

\[
\frac{8 + 16 + 43}{5}
\]

v. \([I]\) an up-and-in knock-in binary put with upper barrier of 105 and strike 110.

\[
\frac{2}{5}
\]
(b) Consider the interest rate tree shown diagram below \((r_u > r_d)\) – each time step is 1-year.

The rates correspond to effective discounting – e.g. discounting over the first period is \(1/(1+r)\). You are told that a contingent claim maturing at \(t = 1\) paying 1 if the interest rates rise and 0 if they drop has value \(C_0 = \frac{1}{4} \frac{1}{1+r}\). Derive an expression for the value of a 2-year zero coupon bond in terms of \(r\), \(r_u\) and \(r_d\) only.

\[
C_0 = \begin{cases} 
1 & \text{if } r_u \\
0 & \text{if } r_d 
\end{cases} 
\Rightarrow C_0 = \frac{q}{1+r} \Rightarrow q = \frac{1}{4} \frac{1}{1+r} 
\Rightarrow q = \frac{1}{4} 
\]

\[
P_0(2) = \begin{cases} \frac{1}{1+r} & \text{if } r_u \\
\frac{1}{1+r} & \text{if } r_d 
\end{cases} 
\Rightarrow P_0(2) = \frac{1}{1+r} \left[ \frac{q}{1+r} + \frac{1-q}{1+r} \right] = \frac{1}{1+r} \left[ \frac{1}{4(1+r_u)} + \frac{3}{4(1+r_d)} \right] 
\]
5. Assume an equity price \( S_t \) is modeled as in the Black-Scholes model (i.e. the limiting case of the CRR model as \( \Delta t \downarrow 0 \) and interest rates are constant at \( r \)). For each of the following, write your answers terms of \( \Phi(x) \triangleq \mathbb{Q}(Z < x) \) where \( Z \) is a standard normal random variable under the risk-neutral measure \( \mathbb{Q} \).

(a) [5] **Derive** an expression for the \((t = 0)\) price of an option with \( T \)-maturity payoff

\[
\varphi = S_T \mathbb{1}_{S_T < K}.
\]

Here \( K \) is a constant and, as usual, \( \mathbb{1}_\omega \) is the indicator function of the event \( \omega \), i.e. equals 1 if \( \omega \) occurs and 0 otherwise.

\[
V_0 = e^{-rT} \mathbb{E}_\mathbb{Q}^\mathbb{Q} \left[ \varphi \right] = e^{-rT} \mathbb{E}_\mathbb{Q}^\mathbb{Q} \left[ S_T \mathbb{1}_{S_T < K} \right]
\]

and \( S_T \overset{\Delta}{=} S_0 \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} \, z \right), \quad z \sim \mathcal{N}(0, 1) \)

\[
= e^{-rT} \mathbb{E}_\mathbb{Q}^\mathbb{Q} \left[ S_T \mathbb{1}_{S_T < K} \right] = \mathbb{E}_\mathbb{Q}^\mathbb{Q} \left[ S_0 \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} \, z \right) \times \mathbb{1}_{z < 3^*} \right]
\]

where \( 3^* = \frac{\ln(K/S_0) - (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \)

\[
\Rightarrow \mathbb{E} = S_0 \exp \left( r - \frac{1}{2} \sigma^2 \right) T \mathbb{E}_\mathbb{Q}^\mathbb{Q} \left[ e^{\sigma \sqrt{T} \, z} \mathbb{1}_{z < 3^*} \right]
\]
\[ f = \int_{-\infty}^{\infty} e^{\sigma \sqrt{T} z} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}} \, dz \]

\[ = \int_{-\infty}^{\infty} e^{\sigma \sqrt{T} z - \frac{1}{2} z^2} \frac{dz}{\sqrt{2\pi}} \]

\[ = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - \sigma \sqrt{T})^2 + \frac{1}{2} \sigma^2 T} \frac{dz}{\sqrt{2\pi}} \]

\[ = e^{\frac{1}{2} \sigma^2 T} \int_{-\infty}^{\infty} e^{-\frac{1}{2} y^2} \, dy \]

\[ = e^{\frac{1}{2} \sigma^2 T} \Phi \left( + 3^* - \sigma \sqrt{T} \right) \]

\[ \Rightarrow \ E = S_0 e^{\Gamma T} \Phi \left( 3^* - \sigma \sqrt{T} \right) \]

\[ \Rightarrow \ V_0 = S_0 \Phi \left( - \frac{\ln (E_0 / \eta) + (r - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} - \sigma \sqrt{T} \right) \]

\[ = S_0 \Phi \left( - \frac{\ln (E_0 / \eta) + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \]
(b) Derive an expression for the \((t = 0)\) price of a forward starting option with \(T\)-payoff

\[
\varphi = \max (S_T, \alpha S_U) - S_V.
\]

Here, \(0 < V < U < T, \; \alpha > 0\) is a constant.

\[
\begin{align*}
\text{by} & \quad c_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ \varphi \right] \\
& = e^{-rT} \left( \mathbb{E}^{\mathbb{Q}} \left[ \max (S_T, \alpha S_U) \right] - \mathbb{E}^{\mathbb{Q}} \left[ S_U \right] \right) \\
& = e^{-rT} \frac{\mathbb{E}^{\mathbb{Q}} \left[ \max (S_T, \alpha S_U) \right] - e^{-rT} \cdot e^{rV} S_o}{h} \\
& = \mathbb{E}^{\mathbb{Q}} \left[ \max (S_T, \alpha S_U) \right] - \mathbb{E}^{\mathbb{Q}} \left[ S_U \right] \\
& = \mathbb{E}^{\mathbb{Q}} \left[ S_T \right] - \mathbb{E}^{\mathbb{Q}} \left[ \alpha S_U \right] \\
& = \mathbb{E}^{\mathbb{Q}} \left[ S_T \right] - \mathbb{E}^{\mathbb{Q}} \left[ \alpha S_U \right] \\
& = \varphi
\end{align*}
\]

Now

\[
S_T = S_U e^\left( (r - \frac{1}{2} \sigma^2) (T - t) + \sigma \sqrt{T - t} Z \right), \quad Z \sim \mathcal{N}(0,1)
\]

\[
\Rightarrow g_i = \mathbb{E}^{\mathbb{Q}} \left[ S_T \right] \quad \text{s.t.} \quad S_T \geq \alpha S_U \mid S_U = S_U e^{(r - \frac{1}{2} \sigma^2) (T - t) + \sigma \sqrt{T - t} Z} = \alpha S_U
\]

where \(3^*\) s.t.

\[
S_U e^{(r - \frac{1}{2} \sigma^2) (T - t) + \sigma \sqrt{T - t} Z} = \alpha S_U
\]

\[
\Rightarrow 3^* = \frac{\ln \alpha - (r - \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}}
\]

\[
\Rightarrow \exists Z \in \mathcal{N}(0,1)
\]

\(\Rightarrow\) independent of \(S_U\)!
\[ g_1 = \sum_{\nu} e^{(\nu + \frac{1}{2} \nu^2) \tau} \int_{3^*}^{+\infty} e^{\nu \bar{F}_3} e^{-\frac{1}{2} \nu^2} \frac{d\nu}{\sqrt{2\pi}} \]

\[ = \sum_{\nu} e^{(\nu - \frac{1}{2} \nu^2) \tau} \int_{3^*}^{+\infty} e^{-t (3^* - \nu^2 \nu^2)} \frac{d\nu}{\sqrt{2\pi}} \]

\[ = \sum_{\nu} e^{\nu \tau} \Phi (-3^* + \nu^2 \nu^2) \]

\[ g_2 = \alpha \sum_{\nu} \left[ \mathbb{P} \left( Z < \nu \right) \right] \]

\[ = \alpha \sum_{\nu} \mathbb{P} \left( Z < \nu \right) \]

\[ = \alpha \sum_{\nu} \Phi (\nu) \]

\[ \Rightarrow \phi = \left[ \sum_{\nu} e^{\nu \tau} \Phi (-3^* + \nu^2 \nu^2) + \alpha \sum_{\nu} \Phi (\nu) \right] \]

\[ = S_0 e^{\nu \tau} \left[ e^{-t \nu} \Phi (-3^* + \nu^2 \nu^2) + \alpha \Phi (\nu) \right] \]

\[ \Rightarrow \phi = S_0 \left( \Phi (-3^* + \nu^2 \nu^2) + e^{-t (\nu - \alpha)} \Phi (\nu) \right) \]

\[ - e^{-t (\nu - \alpha)} S_0 \]
6. Consider the (discrete) model of short rate of interest \( r_n \) (at time step \( n, t_n = n \Delta t \)) given recursively by

\[
\text{independent} \quad r_n = r_{n-1} + \sigma \sqrt{\Delta t} x_n
\]

where \( x_n \) are \( \mathcal{Q} \) (±1) Bernoulli r.v. with \( \mathcal{Q}(x_k = 1) = \frac{1}{2} \).

(a) Determine \( \theta \) such that \( r_n \) has a drift of \( \theta_{n-1} \), i.e. such that \( \mathbb{E}^\mathcal{Q}[r_n - r_{n-1}] = \theta_{n-1} \Delta t \) and find the limiting distribution of \( r_T \) where \( T = N \Delta t \) as \( N \to +\infty \).

\[
\mathbb{E}^\mathcal{Q}[r_T] = \sigma \sqrt{\Delta t} \left( 1 + \theta_{n-1} \Delta t \right)
\]

\[
\sigma \sqrt{\Delta t} (2 q_n - 1) = \theta_{n-1} \Delta t
\]

\[
\Rightarrow q_n = \frac{1}{2} \left[ 1 + \frac{\theta_{n-1} \Delta t}{\sigma} \right]
\]

Clearly, by CLT we'll have \( r_T \overset{\mathcal{D}}{\to} \mathcal{N}(m, \nu) \), so need \( m \) and \( \nu \).

\[
\mathbb{E}^\mathcal{Q}[r_T] = \sigma \sqrt{\Delta t} \sum_{n=1}^{N} \mathbb{E}^\mathcal{Q}[x_n]
\]

\[
= \sum_{n=1}^{N} \theta_{n-1} \Delta t \overset{N \to +\infty}{\to} \int_{0}^{T} \theta_{t} \, ds = m
\]

\[
\mathbb{V}^\mathcal{Q}[r_T] = \sigma^2 \Delta t \sum_{n=1}^{N} \mathbb{V}^\mathcal{Q}[x_n]
\]

\[
= \sigma^2 \Delta t \left( N - \left( \frac{N}{2} \theta_{n-1} \Delta t \right) \Delta t \right) \to \sigma^2 T
\]

since \( x_1, x_2, \ldots \) are independent.

\[
\mathbb{V}^\mathcal{Q}[x_n] = \theta_{n-1} \Delta t \left( 1 - (\theta_{n-1} \Delta t)^2 \right)
\]

\[
\Rightarrow \mathbb{V}^\mathcal{Q}[r_T] = \sigma^2 \Delta t \left( N - \left( \frac{N}{2} \theta_{n-1} \Delta t \right) \Delta t \right) \to \sigma^2 T
\]
(b) \[ \lim_{N \to +\infty} \text{Joint distribution of } r_T \text{ and } I_T = \int_0^T r_s \, ds \text{ where } T = N \Delta t. \]

\[
I_T = \sum_{n=1}^{N} \sum_{m=1}^{n-1} \frac{n-1}{n} x_m \sigma \Delta t^{3/2}
\]

\[
\text{so } \mathbb{E}[I_T^2] = \sum_{n=1}^{N} \sum_{m=1}^{n-1} \frac{n-1}{n} (2q-1) \sigma \Delta t^{3/2}
\]

\[
= \sum_{n=1}^{N} \sum_{m=1}^{n-1} \Theta_{tn-1} \Delta t^2 \rightarrow \int_0^T \int_0^T \Theta_{t} \, du \, ds
\]

Four \( \mathbb{E}[I_T^2] \) cases direct:

\[
\sum_{n=1}^{N} \sum_{m=1}^{n-1} x_m = 0
\]

\[
= (N-1)x_r + (N-2)x_r + \ldots + x_r
\]

\[
\Rightarrow \mathbb{V}[I_T] = \sigma^2 \Delta t \sum_{n=1}^{N-1} (N-n) \mathbb{V}[x_n] = \sum_{n=1}^{N-1} (N-n) \sigma^2 \Delta t^3
\]

\[
\text{are independent.}
\]

\[
\text{now } \mathbb{V}[x_n] = 1 - (\Theta_{tn-1} \Delta t)^2
\]

\[
\Rightarrow \mathbb{V}[I_T] \to \lim_{N \to \infty} \sigma^2 \Delta t^3 \sum_{n=1}^{N-1} (N-n) = \lim_{N \to \infty} \frac{\sigma^2 \Delta t^3}{N} \frac{(2N-1)(N-1)N}{6}
\]

\[
= \frac{\sigma^2 T^3}{3}
\]
\[ \text{next week (CT: } \Gamma_T, I_T) \]

\[ = \mathbb{E} \left[ \sum_{n=1}^{N} x_n \sigma \Delta t, \sum_{n=1}^{N-1} (N-n) x_n \sigma \Delta t^{3/2} \right] \]

\[ = \sigma^2 \Delta t^2 \sum_{n=1}^{N-1} \mathbb{E}[x_n, x_n] (N-n) \]

\[ = \sigma^2 \Delta t^2 \sum_{n=1}^{N-1} (1 - \Theta_{tn-1} \Delta t) (N-n) \]

\[ = \sigma^2 \Delta t^2 \cdot \frac{N(N-1)}{2} \cdot e - \sigma^2 \Delta t^2 \sum_{n=1}^{N-1} (N-n) \Theta_{tn-1} \Delta t \]

\[ \Rightarrow \frac{\sigma^2 \Delta t^2}{2} \]

\[
\begin{pmatrix}
\Gamma_T \\
I_T
\end{pmatrix} \sim \mathcal{N}
\begin{pmatrix}
\left( \int_0^T \Theta_u du \right) \sum_{u=0}^T \Theta_u du & \int_0^T \sum_{u=0}^T \Theta_u du \Theta_u du \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\sigma^2 T}{2} & \frac{\sigma^2 T^2}{2} \\
\frac{\sigma^2 T^2}{2} & \frac{\sigma^2 T^3}{3}
\end{pmatrix}
\]

by CLT.