1. Consider each of the following options:

(a) digital call struck at 100
(b) digital put struck at 100
(c) put struck at 100
(d) call struck at 100
(e) strangle struck at 100
(f) straddle with \( K_1 = 95, K_2 = 115 \)
(g) bull spread with \( K_1 = 95, K_2 = 115 \)

Use the portfolio.xls file to explore the sensitive of prices, Deltas and Gammas to \( T, \sigma, \) and \( r \).

2. Derive the price, delta and gamma for an asset-or-nothing call option (which pays \( S_T \) if \( S_T > K \) at maturity \( T \), and pays 0 otherwise) using the Black-Scholes model. Plot the price, delta and gamma as a function of the spot price with the following parameters: \( S_0 = 1, K = 1, \sigma = 20\%, r = 2\% \) for maturities of \( T = 1/51, 1/12, 1/2 \) and 1 year.

3. Using the Black-Scholes model, determine the price, the delta and the gamma for all times \( t \in [0, T) \) of the following European options with payoffs at time \( T > 0 \):

(a) A forward-start digital call option, which pays 1 at \( T \) if the asset price at maturity is above a percentage \( \alpha \) of the asset price at time \( U \) (where \( t < U < T \)). That is, \( \phi = I(S_T > \alpha S_U) \).

(b) ** A forward-start asset-or-nothing option which pays the asset at \( T \) if the asset price at maturity is above a percentage \( \alpha \) of the asset price at time \( U \) (where \( t < U < T \)). That is, \( \phi = S_T I(S_T > \alpha S_U) \).

(c) A call option (maturing at \( V \)) on a forward-start asset-or-nothing option. The embedded forward-start asset-or-nothing option pays the asset at \( T > V \) if the asset price at \( T \) is above a percentage \( \alpha \) of the asset price at time \( U \) (where \( V < U < T \)). The strike of the call option is \( K \).

4. Suppose that the price of a stock is modeled as follows:

\[
\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t
\]
where $\mu_t$ and $\sigma_t$ are functions only of time and where $W_t$ is a $\mathbb{P}$-Wiener process. Furthermore, assume that the risk-free interest rate $r_t$ is function only of time. Determine the price, the delta and the gamma for each of the following options:

(a) call option maturing at $T$ strike of $K$.
(b) forward starting put option with strike set to $\alpha S_U$ at time $U$ and maturing at $T$.

5. Suppose that interest rates follow the Ho-Lee model:

$$dr_t = \theta_t dt + \sigma_d W_t$$

where $\alpha_t$ is a deterministic function of time and $W_t$ is a $Q$-Wiener process. Determine each of the following:

(a) ** Bond price at time $t$ of maturity $T$.
(b) The SDE which the bond price satisfies in terms of $W_t$.
(c) The choice of $\theta_t$ which makes the model prices equal the market prices $P^*_t(T)$.

6. Suppose that two traded stocks have price processes $X_t$ and $Y_t$. Assume they are jointly GBMs, i.e.

$$\frac{dX_t}{X_t} = \mu_x dt + \sigma_x dW^x_t, \quad \frac{dY_t}{Y_t} = \mu_y dt + \sigma_y dW^y_t,$$

where $X_t$ and $Y_t$ are correlated standard Brownian motions under the $\mathbb{P}$-measure with correlation $\rho$. Consider a contingent claim $f$ written on the two assets with payoff $\varphi(X_T,Y_T)$ at time $T$.

(a) Use a dynamic hedging argument to demonstrate that to avoid arbitrage, the price of $f$ must satisfy the following PDE:

$$\left\{ \begin{array}{l}
\frac{\partial f}{\partial t} + r x \frac{\partial f}{\partial x} + r y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2_x x^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \sigma^2_y y^2 \frac{\partial^2 f}{\partial y^2} + \rho \sigma_x \sigma_y x y \frac{\partial f}{\partial x \partial y} 
\end{array} \right\} f(t,x,y) = r f(T,x,y) = \varphi(x,y).$$

(b) Suppose that the payoff is homogenous, so that $\varphi(x,y) = y g(x/y)$ for some function $g$. An example of such a payoff is the payoff from an exchange option which would have $\varphi(x,y) = (x - y)_+$. By assuming that $f(t,x,y) = y h(t,x/y)$, find the PDE which $h$ satisfies and show that the price $f$ can be written in the form

$$f(X_t,Y_t) = Y_t \mathbb{E}^Q_t[g(U_T)]$$

where, $U_t = X_t/Y_t$ and $X_t$ satisfies an SDE of the form

$$\frac{dU_t}{U_t} = \sigma_U dW^*_t,$$

for some constant $\sigma_U$ and $W^*_t$ a $Q^*$ Brownian motion.