Stochastic Methods for Actuarial Science - Problem Set #2
due Nov, 17 at 2pm.
Hand in only questions marked with **.
Grad students also hand in questions marked with ++

1. Suppose that $W_t$ and $Z_t$ are correlated Wiener processes with instantaneous correlation \( \rho \). Find the mean and variance of each of the following random variables (\( T \) is fixed and positive):
   
   (a) \( W_1 + W_2 + \ldots + W_N \) for \( N \in \mathbb{N} \).
   
   (b) \( \exp\{W_T\} \).
   
   (c) \( aW_T + bZ_T \).
   
   (d) \( \exp\{aW_T + bZ_T\} \).
   
   (e) \( W_T Z_T \).
   
   (f) \( \int_0^T W_s Z_s \, ds \).

2. Suppose that $W_t$ and $Z_t$ are standard correlated Wiener processes with instantaneous correlation \( \rho \). Find the SDE which \( X_t \) solves:
   
   (a) \( X_t = W_t^n \)
   
   (b) \( X_t = \cos(W_t) \)
   
   (c) \( X_t = W_t Z_t \)
   
   (d) \( X_t = \exp(\alpha Z_t + \beta W_t) \)

3. In the following \( W_t \) and \( Z_t \) denote correlated Wiener processes with instantaneous correlation \( \rho \). For each of the following (i) compute the mean and variance of \( Y_t \) and (ii) derive an integration by parts formulae:
   
   (a) \( Y_t = \int_0^t W_s \, dW_s \).
   
   (b) \( Y_t = \int_0^t (W_s)^2 \, dW_s \).
   
   (c) \( Y_t = \int_0^t s W_s \, dW_s \).
   
   (d) \( Y_t = \int_0^t s W_s^2 \, dW_s \).
   
   (e) \( Y_t = \int_0^t s^2 W_s^2 \, dW_s \).
   
   (f) \( Y_t = \int_0^t e^{-W_s} \, dW_s \).
   
   (g) \( \exp\{Y_t\} \).
   
   (h) \( Y_t = \int_0^t e^{-W_s} \, dW_s \).
   
   (i) \( \int_0^t s \, e^{-W_s} \, dW_s \).
   
   (j) \( \int_0^t s \, e^{-W_s} \, dW_s \).
   
   (k) \( \int_0^t s \, e^{-W_s} \, dW_s \).
   
   (l) \( \int_0^t s \, e^{-W_s} \, dW_s \).

   **\( \int_0^t s \, e^{-W_s} \, dW_s \).**
   
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4. Prove the following from first principals, i.e. using the fundamental definition of the Ito integral (Note: \(W_t\) and \(Z_t\) denote correlated Wiener processes with instantaneous correlation \(\rho\)).:

(a) \[ \int W_s^2 dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s ds \]

(b) \[ [++] \int_0^t W_s dZ_s + \int_0^t Z_s dW_s = W_t Z_t - \rho t \]

5. Assume that the prices of two stocks \(S_t\) and \(U_t\) satisfy the following coupled SDEs:

\[
\begin{align*}
\frac{dU_t}{U_t} &= a_t \ dt + b_t \ dW_t^U, \\
\frac{dS_t}{S_t} &= c_t \ dt + d_t \ dW_t^S
\end{align*}
\]

(1)

where \(W_t^U\) and \(W_t^S\) denote two correlated Wiener processes with constant instantaneous correlation \(\rho\) and \(a_t, b_t, c_t, d_t\) are deterministic functions only of time \(t\).

(a) Compute each of the following:

i. \(d \ln(S_t)\)

ii. \(d(S_t U_t)\)

iii. \(d(S_t / U_t)\)

iv. \([**]\) \(d((S_t)^\alpha (U_t)^\beta)\) for \(\alpha \neq 0; \beta \neq 0\).

(b) Solve the system of SDEs (1).

(c) What is the distribution of \(Y = U_T S_T\) for a fixed \(T > 0\)?

6. Suppose \(X_t\) satisfies the SDE:

\[
\begin{align*}
\frac{dX_t}{X_t} &= \kappa(\theta - X_t) \ dt + \sigma \ dW_t \,, \qquad X_0 = x.
\end{align*}
\]

show that the following is a solution

\[
X_0 = \theta + (x - \theta) e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-u)} dW_u
\]