Move based hedging

\[ \Delta_k \text{ units of } S_{t_k} \]

\[ t_{k-1} \quad t_k \]

\[ \Delta_{k-1} \rightarrow \Delta_k \]

\[ S_{k-1} \rightarrow S_k \]

\[ f \text{ in bank at } t_k : \quad M_{k-1} e^{\Delta t_k} \]

\[ @ t_k : \quad M_k = M_{k-1} e^{\Delta t_k} - (\Delta_k - \Delta_{k-1}) S_k \]

\[ \text{trans cost} \]

---

Move based: times at which you trade \( t_k \) are random... but determined by \( \Delta \)
\[ \Delta_t = \partial_s y_t \]

\[ g(t, s) = g(t, s_t) + (s - s_t) \partial_s g(t, s_t) + \frac{1}{2} (s - s_t)^2 \partial_{ss} g(t, s_t) + \ldots \]

\[ \Gamma_t = \partial_{ss} g(t, s_t) \text{ option's gamma} \]

\[ \Delta_t - \text{Gamma Hedging} \]

\[ \alpha_t \text{ write } y_t S_t \]

\[ \beta_t \text{ write } y_t M_t \]
\[ V(t, s) = \alpha_t \Delta h(t, s_t) + \beta_t \partial s h(t, s_t) + r_t h(t, s) \]

\[ \Delta h(t, s_t) + (s - s_t) \frac{\partial s h(t, s_t)}{\partial s} \]

\[ \frac{1}{2} (s - s_t)^2 \frac{\partial^2 s h(t, s_t)}{\partial s^2} + \ldots \]

\[ \Delta g(t, s_t) + (s - s_t) \frac{\partial s g(t, s_t)}{\partial s} \]

\[ \frac{1}{2} (s - s_t)^2 \frac{\partial^2 s g(t, s_t)}{\partial s^2} + \ldots \]
\[ \gamma_t = \frac{\Gamma_t^g}{\Gamma_t^b} \]

\[ \alpha_t = \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^h} \Delta_t^h \]

\[ v = \alpha_t s + \beta_t M_t + \gamma_t h(t, s) \]

\[ \Delta_t^v = \alpha_t + \gamma_t \frac{\partial s}{\partial s} h(t, s) = \Delta_t^g \]

\[ \Gamma_t^v = \Gamma_t^g - \frac{\partial s}{\partial s} h(t, s) = \Gamma_t^g \]

If \( t = 0 \) sold \( q \) get \( q_0 \),

need \( \alpha_0 = \Delta_0^g - \frac{\Gamma_0^g}{\Gamma_0^h} \Delta_0^h \) if \( S \)

\[ \gamma_0 = \frac{\Gamma_0^g}{\Gamma_0^h} \] if \( h \)

in bank \( M_0 = g_0 - \alpha_0 S_0 - \gamma_0 h_0 \)
\[ t = t_1 : \]

\[ M_0 \rightarrow M_0 e^{\Delta t_1} \]

\[ \alpha_0 \alpha_f^f S \rightarrow \alpha_0 \alpha_f^f S \]

\[ (\alpha_0 S_0) \]

\[ \gamma_0 \gamma_f^f h \rightarrow \gamma_0 \gamma_f^f h \]

\[ (\gamma_0 h_0) \]

\[ t = t_0 \]

\[ t = t_1 \]

\[ \text{(P) in general} \]

\[ M_{N+1} = M_{N+1} e^{\Delta t_{N+1}} - (\alpha_{N+1} - \alpha_{N+1}^f) S_{N+1} \]

\[ - (\gamma_{N+1} - \gamma_{N+1}^f) h_{N+1} \]

\[ P_{N+1} = M_{N+1} e^{\Delta t_{N+1}} + \alpha_{N+1} S_{N+1} + \gamma_{N+1} h_{N+1} - \phi_{N+1}(S_T) \]

\[ \text{Vega is sensitivity to vol:} \]

\[ V_t = \partial \gamma g(t, S_t) \]
\[ V_t = \partial_{\tau} g(t, s_t) \]
Options on Dividend Paying Assets

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4:09 PM

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Options on Asset with Dividends

\( S_t \) pays dividends \( \delta S_t \, dt \) at \( t \).

\[
\frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t \quad \text{is risky asset} \quad \alpha_t
\]

\( \beta_t \) bank asset \( \frac{dB_t}{B_t} = r \, B_t \, dt \quad \beta_t 
\]

value a claim on \( S_t \), \( g_t = g(t, S_t) \)
\[
g \in C^1,2
\]

\[
V_t = \alpha_t S_t + \beta_t B_t - g_t
\]

\[
V_0 = 0 \quad \text{start with nothing}
\]

\[
dV_t = \alpha_t \, dS_t + \beta_t \, dB_t - dg_t + \alpha_t \delta S_t \, dt
\]

self-financing

\[
= \alpha_t \left( \mu S_t \, dt + \sigma S_t \, dW_t \right)
\]

\[
+ \beta_t \, r B_t \, dt
\]

\[
- \left[ \left( \frac{\partial g_t}{\partial t} + \mu \, g_t + \frac{1}{2} \sigma^2 S_t \, \delta g_t \right) \right] \, dt
\]
$$\alpha_t = \partial_s g_t$$

$$\Rightarrow dV_t = \int \left( \alpha_t (u + \delta) S_t + \beta_t B_t - (\sigma_t + \delta) g_t \right) dt$$

since 

$$dV_t = \{ \cdot \} dt \quad (i.e., \frac{dV}{dt} \text{ is predictable})$$

$$\Rightarrow \{ \cdot \} = 0 \quad \text{to avoid arbitrage}$$

$$\Rightarrow dV_t = 0 \quad \text{since } V_0 = 0 \Rightarrow V_t = 0$$

$$\Rightarrow \alpha_t S_t + \beta_t B_t - g_t = 0$$

$$\Rightarrow \beta_t = g_t^{-1} \left( g_t - \alpha_t S_t \right)$$

now set $$\alpha_t + \beta_t$$ into $$\{ \cdot \} = 0$$

$$\Rightarrow \partial_s g_t \left( u + \delta \right) S_t + r \left( g_t - \partial_s g_t S_t \right)$$

$$- \left( \partial_t g_t + u S_t \partial_s g_t + \frac{1}{2} \sigma^2 S_t \partial_{ss} g_t \right) = 0$$

$$\Rightarrow \partial_t g_t + (r - \delta) S_t \partial_s g_t + \frac{1}{2} \sigma^2 S_t \partial_{ss} g_t = r g_t$$
\[ \frac{\partial g}{\partial t} + \left( r - \delta \right) S \frac{\partial g}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} g(t, S) = r g(t, S) \]

\[ g(T, S) = q(S) \]

---

**Forward price of an asset**

\[ F_k(T) = \mathbb{E}^Q \left[ e^{-r(T-t)} (S_T - K) \right] \]

Let signing \( F_{t_0}(T) = 0 \)

Sets \( \mathbb{N} \)

\[ \Rightarrow \mathbb{N} = \mathbb{E}^Q \left[ S_T \mid S_{t_0} \right] \]

\[ = S_{t_0} e^{r(T-t)} \]

\( F_k(T) \) is the strike in a forward contract if contract is signed on day \( t_0 \).

\[ F_k(T) = \mathbb{E}^Q \left[ S_T \mid S_{t_0} \right] = S_{t_0} e^{r(T-t)} \]
F_t (T) = E \left[ S_T \mid S_t \right] = S_t e^{r(T-t)}

Futures contracts are like forward contracts except the strike is adjusted each day and equal the forward price \( F_t (T) \), called the futures price.

\[ $2 \rightarrow 1.8 \rightarrow 1.9 \rightarrow 1.85 \]

\[ -0.2 \quad 10.1 \quad -0.05 \]

Costs nothing to enter/leave the contract.

But you pay the losses to reap the gains.
\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]

\[ F_t(T) = S_t e^{r(T-t)} \]

\[ dF_t(T) = dS_t e^{r(T-t)} - rS_t e^{r(T-t)} dt \]

\[ \frac{dF_t(T)}{F_t(T)} = (\mu - r) dt + \sigma dW_t \quad e^{-\alpha t} \]

\[ dB_t = rB_t dt \quad (\beta E) \]

- Claim: \( g_t = g(t, F_t(T)) \) \( g_{T_0} \sim \mathcal{N}(F_{T_0}(T)) \) \( \sim N \)

\[ V_t = \beta_t B_t - g_t \]

\[ dV_t = \beta_t dB_t - dg_t + \alpha_t dF_t(T) \quad \text{Self-financing} \]

\[ = \beta_t rB_t dt - \left[ (\beta_t + (a - r) F_t) \beta_t + \frac{1}{2} \sigma^2 F_t^2 \partial^2 F_t \right] g_t \]
\[ \alpha_t = \partial_F g(t, F_t) \]

\[ \Rightarrow \quad dV_t = \frac{1}{2} \cdot \frac{1}{2} \cdot dt \Rightarrow \quad \frac{1}{2} = 0 \quad \Rightarrow \quad dV_t \text{ is predictable} \]

\[ \therefore \quad \beta_t B_t - g_t = 0 \quad \Rightarrow \quad \beta_t = B_t^{-1} g_t \]

\[ \log \quad \Rightarrow \quad \log q_t - \left( \partial_t g_t + (\kappa - r) F_t \partial_F g_t + \frac{1}{2} \sigma^2 \partial_{FF} g_t \right) \]

\[ + \partial_F g_t (\mu - r) F_t \]

\[ = 0 \]

\[ \Rightarrow \quad \partial_t g_t + \frac{1}{2} \sigma^2 F_t \partial_{FF} g_t = \log q_t \]

\[ \text{a penalty of } F_t \]
\[
\begin{aligned}
\frac{\partial}{\partial t} g(t, F) + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2}{\partial F^2} g(t, F) &= r g(t, F) \\
q(T_0, F) &= \phi(F)
\end{aligned}
\]