UNIVERSITY OF TORONTO

Faculty of Arts and Science

Final Examination December 14, 2009

ACT460H1 F / STA2502H F

DURATION - 3 hours

EXAMINER: Prof. S. Jaimungal

LAST NAME: ________________________________

FIRST NAME: ________________________________

STUDENT #: ________________________________

Each question is worth 10 points.

Please write clearly!

AIDS: Calculators Only

PLEASE HAND IN AND WRITE YOUR ANSWERS IN THIS BOOKLET.

Exam Contains: 30 pages
• Normal cdf: $\Phi(x) := \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$.

• Moments of Normals: If $Z$ is a normal r.v. with mean 0 and variance 1, then the m.g.f. is $\mathbb{E}[e^{aZ}] = e^{\frac{1}{2}a^2}$ and the fourth moment is $\mathbb{E}[Z^4] = 3$.

• 2-D Ito’s Lemma: If $W_t$ and $Z_t$ are standard correlated Brownian motions with $d[W, Z] = \rho dt$ and $X_t$ and $Y_t$ satisfy the SDEs:

$$dX_t = \mu^X_t\,dt + \sigma^X_t\,dW_t, \quad dY_t = \mu^Y_t\,dt + \sigma^Y_t\,dZ_t$$

and $U_t = f(X_t, Y_t, t)$, where $f(x, y, t)$ is twice differentiable in $x$ and $y$ and once differentiable in $t$, then

$$dU_t = \partial_x f(X_t, Y_t, t)dX_t + \partial_y f(X_t, Y_t, t)dY_t$$

$$+ \left(\frac{1}{2}(\sigma^X_t)^2 \partial_{xx} + \frac{1}{2}(\sigma^Y_t)^2 \partial_{yy} + \rho \sigma^Y_t \sigma^X_t \partial_{xy}\right) f(X_t, Y_t, t) \ dt$$

$$= \left(\partial_t + \mu^X_t \partial_x + \mu^Y_t \partial_y + \frac{1}{2}(\sigma^X_t)^2 \partial_{xx} + \frac{1}{2}(\sigma^Y_t)^2 \partial_{yy} + \rho \sigma^Y_t \sigma^X_t \partial_{xy}\right) f(X_t, Y_t, t) \ dt$$

$$+ \sigma^X_t \partial_x f(X_t, Y_t, t) dW_t + \sigma^Y_t \partial_y f(X_t, Y_t, t) dZ_t.$$ 

• Ito’s Isometry: If $W_t$ is a standard Brownian motion, then $\mathbb{E} \left[ \left( \int_0^t g_s \, dW_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t g_s^2 \, ds \right]$ for all $\mathcal{F}_t$-adapted processes $g_t$.

• Feynman-Kac: Suppose a function $f(y, t)$ satisfies the PDE:

$$\begin{cases} 
(\partial_t + a(y, t)\partial_y + \frac{1}{2}b^2(y, t)\partial_{yy}) f(y, t) = c(y, t) f(y, t) \\
f(y, T) = \varphi(y) 
\end{cases}$$

Then $f(y, t)$ admits the unique solution

$$f(y, t) = \mathbb{E}^M[e^{-\int_t^T c(Y_s, s) \, ds} \varphi(Y_T) \mid Y_t = y]$$

where,

$$dY_t = a(Y_t, t) \, dt + b(Y_t, t) \, dW_t$$

and $W_t$ is a $\mathbb{M}$-standard Brownian motion.
1. Briefly explain each of the following concepts:

(a) \([5]\) Arbitrage

(b) \([5]\) A Brownian motion.
2. \(10\) Please indicate true or false (no explanations required).

+2 for correct answer; −0.5 for incorrect answer; 0 for no answer.

(a) \[T\] \[F\]

An economy has the two traded assets shown below. This economy admits an arbitrage.

(b) \[T\] \[F\]

You have sold a put option on XYZ shares and you are simultaneously delta-hedging the position. Suppose that important (unexpected) news arrives declaring poor sales of XYZ products resulting in a drop in share value. You must sell shares of XYZ to maintain your hedge.

(c) \[T\] \[F\]

If \(X_t = \mu t + W_t\) where \(\mu > 0\) and \(W_t\) is a standard Brownian motion, then the variance of \(X_t\) is equal to \(t\).

(d) \[T\] \[F\]

Delta hedging using a move-based approach always outperforms hedging using a time-based approach.

(e) \[T\] \[F\]

Suppose that a call option struck at 10 is selling for 1; while a call option struck at 20 is selling for 2. Both call options have the same maturity. This economy admits an arbitrage.
3. (a) [5] Consider the following portfolio: 4 long puts struck at 1 and one long call struck at 1. Sketch the \textit{delta} of the portfolio (i) at maturity (ii) 1-year from maturity on the same graph below. Label any important points clearly.
(b) [5] Sketch the **gamma** of an **asset-or-nothing call option** struck at 1 (i) at maturity (ii) 1-year from maturity on the same graph below. Label any important points clearly. *[Recall that an asset-or-nothing call has a payoff of $S_T$ if $S_T > K$, otherwise it pays 0.]*
4. Consider a simple two-step binomial model of interest rates in which \( r_0 = R \), and \( r_{n} = r_{n-1} \pm 1\% \).
(Treat these rates as per period discount rates – e.g. discounting over the first period is \( 1/(1 + R) \)).

(a) [5] Determine \( R \) and the risk-neutral branching probabilities over the first period consistent with the following market prices:

- A 1 period zero coupon bond costs $95.
- A 2 period zero coupon bond costs $90.
Blank intentionally. Continue work here...
(b) Suppose that $R = 5\%$ and the risk-neutral branching probabilities are $q = 1/2$.

Consider a European call option on a 3-period zero coupon bond with notional 100. The option matures at $t = 2$ and the strike of the option is 95. Determine the value of the option.
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5. Consider an asset-or-nothing call option in the Black-Scholes model with zero interest rates. Recall that an asset-or-nothing option pays \( \varphi = S_TI(S_T > K) \) at maturity \( T \).

(a) Show that the price of the option is

\[
V(S, t) = S \Phi(d_+) , \quad d_+ = \frac{\ln(S/K)}{\sigma(T-t)^{1/2}} + \frac{1}{2} \sigma(T-t)^{1/2} .
\]
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(b) [5] Confirm that the price satisfies the Black-Scholes partial differential equation.

[Hint: use the fact that \( \Phi''(x) = -x\Phi'(x) \)]
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6. You are given that $W_t$ and $B_t$ are correlated Brownian motions with correlation $\rho$.

   (a) [5] Obtain an integration by parts formula for $\int_0^t W_s^2 dB_s$. 
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(b) Determine the mean and variance of \( X_t = \int_0^t (W_s + B_s) \, dB_s \).
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7. Suppose that two stocks $U_t$ and $V_t$ satisfy the following SDEs:

\[
\frac{dU_t}{U_t} = \alpha \, dt + \sigma \, dX_t, \quad \frac{dV_t}{V_t} = \beta \, dt + \eta \, dY_t,
\]

where $X_t$ and $Y_t$ are $\mathbb{P}$-Wiener processes with correlation $d[X,Y]_t = \rho \, dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free rate is zero.

(a) [5] Determine the SDE which $G_t := U_t/V_t$ satisfies and the distribution of $G_t$ for a fixed $t$ conditional on $G_0$. 
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(b) [5] Determine the price at time \( t = 0 \) of an option which pays

\[
\varphi = \frac{V_S}{U_T} \mathbb{I}(V_S > \gamma)
\]

at the maturity date \( T \) and \( T > S > 0 \). Here, \( \gamma \) is a positive constant.
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8. Suppose that two stocks $X_t$ and $Y_t$ satisfy the following SDEs:

$$\frac{dX_t}{X_t} = \alpha \, dt + \sigma \, dW^X_t, \quad \frac{dY_t}{Y_t} = \beta \, dt + \eta \, dW^Y_t,$$

where $W^X_t$ and $W^Y_t$ are $\mathbb{P}$-Wiener processes with correlation $d[W^X, W^Y]_t = \rho \, dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free interest rate $r$ is constant. Furthermore, an option written on the two stocks has a payoff at maturity of $\varphi(X_T, Y_T)$.

(a) [5] Through a **dynamic hedging argument** (analogous to what we covered in class), **prove** that the price of the option $g(x, y, t)$ satisfies the following PDE:

$$\begin{cases}
(\partial_t + r x \partial_x + r y \partial_y + \frac{1}{2} \sigma^2 x^2 \partial_{xx} + \frac{1}{2} \eta^2 y^2 \partial_{yy} + \rho \sigma \eta xy \partial_{xy}) \, g = r \, g , \\
g(x, y, T) = \varphi(x, y) .
\end{cases}$$
Blank intentionally. Continue work here...
Blank intentionally. Continue work here...
(b) Suppose that \( \varphi(x, y) = x \mathbb{1}(y > \gamma) \). Assume that \( g(x, y, t) \) can be written as \( g(x, y, t) = x f(y, t) \) for some function \( f(y, t) \). Using the PDE from part (a), derive a PDE for \( f(y, t) \) and solve for it using the Feynman-Kac result.
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