UNIVERSITY OF TORONTO

Faculty of Arts and Science

Final Examination, December 10th, 2008

ACT460H1 F / STA2502H F

DURATION - 3 hours

EXAMINER: Prof. S. Jaimungal

COURSE CODE (circle one): ACT460H1 F STA2502H F

LAST NAME: ________________________________

FIRST NAME: ________________________________

STUDENT #: ________________________________

Each question is worth 10 points.

Please write clearly!

AIDS: Calculators Only

Exam Contains: 29 pages

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• Normal cdf: \( \Phi(x) := \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}} \).

• Moments of Normals: If \( Z \) is a normal r.v. with mean 0 and variance 1, then the m.g.f. is \( \mathbb{E}[e^{aZ}] = e^{\frac{1}{2}a^2} \) and the fourth moment is \( \mathbb{E}[Z^4] = 3. \)

• 2-D Ito’s Lemma: If \( W_t \) and \( Z_t \) are standard correlated Brownian motions with \( d[W,Z] = \rho dt \) and \( X_t \) and \( Y_t \) satisfy the SDEs:

\[
\begin{align*}
  dX_t &= \mu_t^X \, dt + \sigma_t^X \, dW_t, \\
  dY_t &= \mu_t^Y \, dt + \sigma_t^Y \, dZ_t
\end{align*}
\]

and \( U_t = f(X_t,Y_t,t) \), where \( f(x,y,t) \) is twice differentiable in \( x \) and \( y \) and once differentiable in \( t \), then

\[
dU_t = \left( \frac{\partial}{\partial t} + \mu_t^X \frac{\partial}{\partial x} + \mu_t^Y \frac{\partial}{\partial y} + \frac{1}{2} \sigma_t^X \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sigma_t^Y \frac{\partial^2}{\partial y^2} + \rho \sigma_t^X \sigma_t^Y \frac{\partial^2}{\partial x \partial y} \right) f(X_t,Y_t,t) \, dt \\
+ \sigma_t^X \frac{\partial}{\partial x} f(X_t,Y_t,t) \, dW_t + \sigma_t^Y \frac{\partial}{\partial y} f(X_t,Y_t,t) \, dZ_t.
\]

• Ito’s Isometry: If \( W_t \) is a standard Brownian motion, then \( \mathbb{E} \left[ \left( \int_0^t g_s \, dW_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t g_s^2 \, ds \right] \) for all \( \mathcal{F}_t \)-adapted processes \( g_t \).

• Ito’s Product and Quotient Rule:

\[
\begin{align*}
  \frac{d(X_t Y_t)}{X_t Y_t} &= \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + \frac{d[X,Y]}{X_t Y_t} ; \\
  \frac{d(X_t/Y_t)}{X_t/Y_t} &= \frac{dX_t}{X_t} - \frac{dY_t}{Y_t} + \frac{d[Y,Y]}{Y_t^2} - \frac{d[X,Y]}{X_t Y_t}.
\end{align*}
\]

• Girsanov’s Theorem: Given two equivalent measures \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \), there exists an \( \mathcal{F}_t \)-adapted vector process \( \tilde{\lambda}_t \) such that if \( \tilde{W}_t \) is a vector of \( \mathbb{P}_1 \)-Wiener process, then \( \tilde{W}_t^* \), satisfying \( d\tilde{W}_t^* = \tilde{\lambda}_t \, dt + d\tilde{W}_t \), is a \( \mathbb{P}_2 \)-Wiener process.
1. Briefly explain each of the following concepts:

   (a) Arbitrage.

   (b) A Brownian motion.
2. [10] Please indicate true or false (no explanations required).
+2 for correct answer; −0.5 for incorrect answer; 0 for no answer.

(a) [T] [F]
In an economy with three tradable assets, it is never possible to replicate contingent claims written on one of the assets.

(b) [T] [F]
You have sold a put option on XYZ shares and you are simultaneously delta-hedging the position. Suppose that important (unexpected) news arrives declaring poor sales of XYZ products resulting in a drop in share value. You must sell shares of XYZ to maintain your hedge.

(c) [T] [F]
If interest rates are modeled as $dr_t = \theta dt + \sigma dW_t$ where $W_t$ is a Brownian motion, then interest rates mean-revert.

(d) [T] [F]
If the real-world evolution of share prices evolves with a vol of 20% and you delta-gamma hedge a put option with a vol of 25% on a daily basis, then the net PnL will be symmetric.

(e) [T] [F]
Suppose that a put option struck at 1 is selling for 0.10; while a put option struck at 2 is selling for 0.2. Both puts have the same maturity. This economy admits an arbitrage.
3. (a) [5] Consider the following portfolio: long put struck at 80 and a long call struck at 100. Sketch the delta of the portfolio (i) at maturity (ii) 1-month from maturity (iii) 1-year from maturity all on the same graph. Label any important points clearly.
(b) [5] Sketch the \textbf{gamma} of a \textbf{digital call} option (i) at maturity (ii) 1-month from maturity (iii) 1-year from maturity all on the same graph. Label any important points clearly. \textit{Recall that a digital call option pays 1 at maturity if the asset price exceeds the strike $K$ otherwise it pays nothing.}
4. Consider a simple two-step binomial model of interest rates in which \( r_0 = 5\% \), and \( r_n = r_{n-1} \pm 1\% \). (Treat these rates as per period discount rates – e.g. discounting over the first period is \( 1/1.05 \)).

(a) Determine the risk-neutral branching probabilities consistent with a market price of 100 for a coupon bearing bond which pays 5 at \( t = 1 \) and 105 at \( t = 2 \).
Blank intentionally. Continue work here...
(b) Suppose that the risk-neutral branching probabilities are $q = 1/2$.

Consider a European call option on a 3-period bond with notional 100. The option matures at $t = 2$ and the strike of the option is 95. Determine the value of the option.
Blank intentionally. Continue work here...
5. Consider a digital call option in the Black-Scholes model with zero interest rates.

(a) Show that the price of the digital call option is

\[ V(S, t) = \Phi(d_-), \quad d_- = \frac{\ln(S/K)}{\sigma(T-t)^{1/2}} - \frac{1}{2}\sigma(T-t)^{1/2}. \]
Blank intentionally. Continue work here...
(b) [5] Confirm that the price satisfies the Black-Scholes partial differential equation.

[Hint: use the fact that $\Phi''(x) = -x\Phi'(x)$]
Blank intentionally. Continue work here...
6. You are given that $W_t$ and $B_t$ are correlated Brownian motions with correlation $\rho$.

(a) [5] Obtain an integration by parts formula for $\int_0^t e^{W_s} dB_s$. 
Blank intentionally. Continue work here...
(b) Determine the mean and variance of \( X_t = \int_0^t W_s dB_s - \int_0^t B_s dW_s \).
Blank intentionally. Continue work here...
7. Suppose that two stocks $U_t$ and $V_t$ satisfy the following SDEs:

$$\frac{dU_t}{U_t} = \alpha dt + \sigma dX_t, \quad \frac{dV_t}{V_t} = \beta dt + \eta dY_t,$$

where $X_t$ and $Y_t$ are $\mathbb{P}$-Wiener processes with correlation $d[X,Y]_t = \rho dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free rate is zero.

Determine the price at time $t = 0$ of an option which pays

$$\varphi = U_T \times 1_{V_S > \gamma}$$

at the maturity date $T$ and $T > S > 0$. Here, $\gamma$ is a constant.
Blank intentionally. Continue work here...
8. \[10\] Prove that
\[
\int_0^t W_s dZ_s + \int_0^t Z_s dW_s = W_t Z_t - \rho t \quad a.s.
\]
Do not use Ito’s lemma, but rather use the fundamental definition of the stochastic integrals.
Blank intentionally. Continue work here...
Blank intentionally. Continue work here...
9. Consider the Vasicek model for the short rate of interest:

\[ dr_t = \kappa(\theta - r_t) \, dt + \sigma \, dW_t \]

where \( W_t \) is a \( \mathbb{Q} \)-Wiener process. The solution to this SDE is

\[ r_s = \theta + (r_t - \theta) \, e^{-\kappa(s-t)} + \sigma \int_t^s e^{-\kappa(s-u)} \, dW_u \quad \text{for} \quad t \leq s. \]

(a) [5] Show that the distribution of \( I_t^T = \int_t^T r_s \, ds \) is normal with mean \( m \) and variance \( v \) with

\[
\begin{align*}
    m &= \theta((T - t) - B(T - t; \kappa)) + B(T - t; \kappa) r_t, \\
    v &= \frac{\sigma^2}{\kappa^2} ((T - t) + B(T - t; 2\kappa) - 2B(T - t; \kappa))
\end{align*}
\]

where, \( B(\tau; \kappa) = \frac{1}{\kappa} (1 - e^{-\kappa\tau}) \).
Blank intentionally. Continue work here...
Blank intentionally. Continue work here...
(b) [5] Show that price of a $T$-maturity bond satisfies the SDE
\[
\frac{dP_t(T)}{P_t(T)} = r_t \, dt - \sigma B(T - t; \kappa) \, dW_t.
\]
Blank intentionally. Continue work here...