1. [10] Please indicate true or false. no explanations required
   -1 for incorrect answer, +2 for correct answer, 0 for blank answer.

(a) [T] [F] 
   All two period, two state (binomial) economies are arbitrage free.
   
   false, e.g. \[ 1 < 1 \, \, 1 < 2 \]

(b) [T] [F]  
   The price of a call option always decreases with increasing volatility.

   False. It increases.

(c) [T] [F] 
   If the branching probabilities are unique, then all contingent claims can be replicated.

   True. Unique \( q^* \)’s \( \Rightarrow \) unique prices \( \Rightarrow \) replication

(d) [T] [F] 
   The risk-neutral return of the short rate of interest in a stochastic interest rate model is equal to \( r \).

   Short rate of interest is not traded so \( \mathbb{E}^*[-r_t] \neq r_t e^r \)

(e) [T] [F] 
   Suppose interest rates are zero. An at-the-money put option is worth 0.80 and a call option with the same strike and maturity is worth 0.75. This economy admits an arbitrage.

   [At-the-money means the strike equals the spot.]

   \( \text{Call} - \text{Put} = \text{Spot} - K = 0 \) since at the
arbitrage
2. Sketch the option price as a function of the current spot-level for maturities of \( T = 0 \), \( T = 1 \) month and \( T = 1 \) year for

(a) \( \{5\} \) digital call option (which pays 1 if \( S > K \) and 0 otherwise).

[draw the three curves on the same graph, clearly label them and any interesting points.]
(b) [5] A portfolio of 4 long puts and 1 long call, both struck at $1.

[draw the three curves on the same graph, clearly label them and any interesting points..]
3. [10] Consider an economy with the two traded assets below. Find the values of $X$ such that the economy is free of arbitrage.

\[
\begin{align*}
100 (1+r) &= 120 q + X (1-q) \\
10 (1+r) &= 20 q + 5 (1-q)
\end{align*}
\]

\[\Rightarrow 10 = \frac{120 q + X (1-q)}{5 + 15 q}\]

\[\Rightarrow 50 + 150 q = (120 - X) q + X\]

\[\Rightarrow (30 + X) q = X - 50\]

\[\Rightarrow q = \frac{X - 50}{30 + X}\]

\[0 < q < 1 \iff X > 50\]

**Method 2:**

Choose $B$ as numerical, so relative price tree is $b$.
\[
10 = 6 \frac{q^a}{x} + \frac{x}{5} (1-q^b) \\
50 = 30 q^b + x (1-q^b) \\
\Rightarrow q^b = \frac{50-x}{30-x}
\]

to avoid arbitrary and \( 0 < q^b < 1 \)

\Rightarrow 0 < \frac{50-x}{30-x} < 1

\textcircled{1} \quad x < 30 \Rightarrow 0 < 50-x < 30-x

\Rightarrow x < 50 \text{ and } 50 < 30 \quad \text{contradiction}

\textcircled{2} \quad x > 30 \Rightarrow 0 > 50-x > 30-x

\Rightarrow \boxed{x > 50}
4. Consider the interest rate tree shown in the diagram below – each time step is 1-year. The rates correspond to effective discounting – e.g. discounting over the first period is $1/(1 + R)$. The probabilities shown are risk-neutral probabilities.

(a) [6] The price of a one-year bond on a notional of $100 is $95.2381. As well, a 2-year coupon bearing bond with coupons of $5 paid every year and notional of $100 is valued at par. Calibrate this model to the market prices, i.e. determine $R$ and $q$ such that the market prices are equal to the model prices.

\[
p(1) = \frac{100}{1 + R} = 95.2381 \Rightarrow R = 5.25\%
\]

\[
p(2) = \left( \frac{105}{1.06} \right) = 104.06
\]

\[
p(2) = \left( \frac{105}{1.04} \right) = 105.96
\]

\[
100 = p(2) = \frac{1}{1.05} \left[ 104.06 q + 105.96 (1 - q) \right]
\]

\[
\Rightarrow 105 = 104.06 q + 105.96 (1 - q)
\]

\[
\Rightarrow q = \frac{0.96}{105.96 - 104.06} = 0.5053
\]
(b) Now assume that \( q = \frac{1}{2} \) and \( R = 5\% \). As well, you can only trade using the 1-year and 2-year zero coupon bonds with notional of $100 (i.e. 1-year zero coupon bond pays $100 at year 1, and the 2 year zero coupon bond pays $100 at year 2).

What is the replication strategy of an option which pays $100 if the interest rate drops to 4%?

\[
\begin{align*}
\alpha & = \frac{100}{95.2381} = 1.0617 \\
\beta & = \frac{100}{96.15} = 1.0319 \\
\end{align*}
\]

Now, let's consider the probability of each event:

\[
P(\text{i}) = \frac{1}{1.05} \left[ \frac{1}{2} \cdot 94.34 + \frac{1}{2} \cdot 96.15 \right] = 90.71
\]

For the zero coupon bond:

\[
0 = \alpha \cdot 100 + \beta \cdot 94.34
\]

And for the 1-year bond:

\[
100 = \alpha \cdot 100 + \beta \cdot 96.15
\]

Solving for \( \beta \):

\[
\beta = \frac{100}{96.15 - 94.34} = 55.25
\]

Solving for \( \alpha \):

\[
\alpha = -\beta \cdot \frac{96.15}{100} = -53.12
\]
5. Assume an equity price $S_t$ is modeled as in the Black-Scholes model (i.e. the limiting case of the CRR model as $\Delta t \downarrow 0$ and interest rates are constant at $r$). For each of the following, write your answers terms of $\Phi(x) \triangleq Q(Z < x)$ where $Z$ is a standard normal random variable under the risk-neutral measure $Q$.

(a) [5] Derive an expression for the $(t = 0)$ price of an option with $T$-maturity payoff

$$\varphi = \min(S_T ; K) .$$

Here $K$ is a constant.

\[ S_T \overset{d}{=} S \exp \left\{ (r - \frac{1}{2} \sigma^2) T + \sigma \sqrt{T} Z \right\} , \quad Z \sim \mathcal{N}(0,1) \]

\[ V = e^{-rT} \mathbb{E}^Q \left[ \min \left( S_T , K \right) \right] \]

Then, \[ \min \left( S_T , K \right) = K \mathbb{1}_{S_T > K} + S_T \mathbb{1}_{S_T \leq K} \]

\[ \therefore V = e^{-rT} \left( K \mathbb{1}_{S_T > K} + \mathbb{E}^Q \left[ S_T \mathbb{1}_{S_T \leq K} \right] \right) \]

\[ \mathbb{E}^Q \left[ S_T \mathbb{1}_{S_T \leq K} \right] = \int_{-\infty}^{\infty} \mathbb{E} \left( \exp \left( (r - \frac{1}{2} \sigma^2) T + \sigma \sqrt{T} Z \right) \mathbb{1}_{Z < z} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \]

\[ \Rightarrow z^* = \frac{\ln \left( S/K \right) + (r - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \]
\[ \begin{align*}
&= S \ e^{(r - \frac{1}{2} \sigma^2) \tau} \int_{-\infty}^{3^{-}} e^{\sigma \sqrt{T} z} - \frac{1}{2} z^2 \ \frac{dz}{\sqrt{2\pi}} \\
&= S \ e^{(r - \frac{1}{2} \sigma^2) \tau} \int_{-\infty}^{3^{+}} e^{- \frac{1}{2} (z - \sigma \sqrt{T})^2 + \frac{1}{2} \sigma^2 \tau} \ \frac{dz}{\sqrt{2\pi}} \\
&= S \ e^{r \tau} \int_{-\infty}^{3^{+} - \sigma \sqrt{T}} e^{- \frac{1}{2} z^2} \ \frac{dz}{\sqrt{2\pi}} \\
&= S \ e^{r \tau} \ \Phi \left( \frac{\ln(S/k) + (r + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{T}} \right) \\
\end{align*} \]

and also,

\[ \begin{align*}
Q (S_T > k) &= Q (Z > 3^*) = \Phi (3^*) \\
&= \Phi \left( \frac{\ln(S/k) + (r - \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{T}} \right) \\
\end{align*} \]

so final answer is:

\[ V = k \ e^{-r \tau} \ \Phi (d_-) + S \ \Phi (-d_+) \]

(b) [5] Derive an expression for the \((t = 0)\) price of a forward start option with \(T\)-maturity payoff

\[ \varphi = \min (S_T \ k \ S_U) . \]

Here, \(0 < U < T\) and \(k\) is a proportionality constant.
\[ V = e^{-rt} \mathbb{E}^{\alpha} \left[ \min (S_T; kS_u) \right] \\
= e^{-rt} \mathbb{E}^{\alpha} \left[ \mathbb{E}^{\alpha} \left[ \min (S_T, kS_u) \mid S_u \right] \right] \\
= e^{-rt} \mathbb{E}^{\alpha} \left[ \left( kS_u e^{-r(T-u)} \Phi(l-) \right. \right. \\
\left. \left. + S_u \Phi(l+) \right) e^{r(T-u)} \right] \\
\]

here \[ l_\pm = \frac{\ln(S_u/kS_u) + (r \pm \frac{1}{2} \sigma^2)(T-u)}{\sigma \sqrt{T-u}} \]
\[ \text{are constants!} \]

\[ \Rightarrow V = \left( k e^{-r(T-u)} \Phi(l-) + \Phi(l+) \right) e^{-ru} \mathbb{E}^{\alpha} \left[ S_u \right] \\
= \left( k e^{-r(T-u)} \Phi(l-) + \Phi(l+) \right) S \]
6. Consider the CRR model of stock prices

\[ S_{n+1} = S_n e^{\sigma \sqrt{\Delta t} x_n} \]

where \( x_1, x_2, \ldots \) are iid r.v. with \( \mathbb{P}(x_1 = +1) = p \) and \( \mathbb{P}(x_1 = -1) = 1 - p \). Interest rates are constant so that the money-market account \( M_t \) evolves as

\[ M_{n+1} = M_n e^{r \Delta t} \]

(a) Prove that under the measure induced by using \( S \) as a numeraire asset (call this measure \( Q_S \)), as \( \Delta t \downarrow 0 \) one has

\[ S_T \overset{d}{=} S \exp\{(r + \frac{1}{2} \sigma^2) T + \sigma \sqrt{T} Z\} \]

where \( Z \overset{d}{\sim} \mathcal{N}(0,1) \).

[Note that the drift is \( r + \frac{1}{2} \sigma^2 \) and NOT \( r - \frac{1}{2} \sigma^2 \) as it is under the risk-neutral measure \( Q \).]

\[ \text{we set as numeraire, so relative price tree is} \]

\[ \begin{array}{c}
\text{ } \\
1/ S < \frac{q_S}{e^{rt} / S} e^{\sigma \sqrt{\Delta t}} \\
\text{ } \\
\text{want } q_S \text{ as } \Delta t \downarrow 0 \ldots \\
\end{array} \]

\[ q_S = \frac{e^{-r \Delta t} - e^{\sigma \sqrt{\Delta t}}}{e^{-\sigma \sqrt{\Delta t}} - e^{\sigma \sqrt{\Delta t}}} \]

\[ \begin{align*}
\text{ } \\
q_S &= \frac{(1 - r \Delta t) - (1 + \sigma \sqrt{\Delta t} + \frac{1}{2} \sigma^2 \Delta t)}{(1 - \sigma \sqrt{\Delta t} + \frac{1}{2} \sigma^2 \Delta t) - (1 + \sigma \sqrt{\Delta t} + \frac{1}{2} \sigma^2 \Delta t) t} \\
&= -\sigma \sqrt{\Delta t} - (r + \frac{1}{2} \sigma^2) \Delta t + \ldots
\end{align*} \]
- 2 \sigma \sqrt{\Delta t} + \ldots \\
= \frac{1}{2} \left[ 1 + \frac{r + \frac{1}{2} \sigma^2 \sqrt{\Delta t}}{\sigma^2} \right]

so here since

\lambda(S_T/S) = \sigma \sqrt{\Delta t} \sum_{m=1}^{n} x_m \\
\rightarrow \mathcal{N} \left( \mu_{max}, \nu_{max} \right) \text{ by CLT}

\frac{x_i}{\lambda^2} \\
\text{IE} \left[ \lambda(S_T/S) \right] = \sigma \sqrt{\Delta t} \nu \text{IE} \left[ x_i \right] \\
= \sigma \sqrt{\Delta t} \nu \left( 2 q_s - 1 \right) \\
= \sigma \sqrt{\Delta t} \nu \left( \frac{r + \frac{1}{2} \sigma^2 \sqrt{\Delta t} + \ldots}{\sigma} \right) \\
\rightarrow (r + \frac{1}{2} \sigma^2) \nu \Delta t

\text{and}

\text{IE} \left[ \left( \frac{\lambda(S_T/S)}{\lambda^2} \right)^2 \right] = \sigma^2 \Delta t \nu \text{IE} \left[ x_i \right] \\
= \sigma^2 \Delta t \nu \left( 1 - (2 q_s - 1)^2 \right) \\
= \sigma^2 \Delta t \nu \left( 1 - \left( \frac{r + \frac{1}{2} \sigma^2 \sqrt{\Delta t} + \ldots}{\sigma} \right)^2 \right) \\
\rightarrow \sigma^2 \nu

\therefore \quad S_T \overset{d}{=} \exp \left( (r + \frac{1}{2} \sigma^2) T + \sigma \sqrt{T} Z \right)
(b) \[ (k - s_T)^+ = (k - s_T) \mathbb{1}_{s_T < k} \]

\[ = k \mathbb{1}_{s_T < k} - s_T \mathbb{1}_{s_T < k} \]

\[ = k \left[ \mathbb{1}_{s_T < k} - s_T \left( \frac{\ln(S/K) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right) \right] \sim \mathcal{N}(0,1) \]

\[ \frac{V_0^A}{S} = \mathbb{E}^{Q} \left[ \frac{V_T^A}{S_T} \right] \]

[Hint: Write the put payoff in terms of a digital option \( K \mathbb{1}_{s_T < k} \) and an asset-or-nothing option \( S_T \mathbb{1}_{s_T < k} \) and value each separately.]
\[
E \left[ \frac{S_T \mathbb{1}_{S_T < \kappa}}{S_T} \right] = E \left[ \mathbb{1}_{S_T < \kappa} \right]
\]

\[
= \mathbb{Q}^S \left( S_T < \kappa \right) = \mathbb{Q}^S \left( \mathcal{E}^{Q^S} < - \frac{\lambda_0 (S/\kappa) + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right)
\]

\[
= \Phi(-d_+)
\]

\[
\Rightarrow V = V^A_v - V^B_v = \kappa e^{-rT} \Phi(-d_-) - \Lambda \Phi(-d_+)
\]