1. [10] Please indicate true or false. no explanations required.
   -1 for incorrect answer, +2 for correct answer, 0 for blank answer.

(a) [T] [F]
   If an economy admits a strategy which costs nothing and at a future time has strictly positive values with probability greater than zero, then this economy admits an arbitrage.
   
   must also have $P(V_t \geq 0) = 1$

(b) [T] [F]
   The price of a put option always increases with volatility.
   
   more uncertainty increases value of the option

(c) [T] [F]
   In a one-period economy, the risk-neutral branching probabilities are always uniquely determined.

   only if number of traded assets equals number of states, and the assets are not redundant.

(d) [T] [F]
   If $S_t$ is the price of a traded stock, then in the Black-Scholes economy, the expected rate of return of $S_t^2$ is equal to $2r + \sigma^2$.

\[
E[S_t^2] = E[e^{2(r - \frac{1}{2}\sigma^2)t} | E[e^{\sigma W} | \sigma = \sigma] = S_0 e^{2(r - \frac{1}{2}\sigma^2)t} e^{\frac{1}{2} \sigma^2 t} = S_0 e^{2r + \sigma^2 t}
\]

(c) Sebastian Jaimungal, 2009
(e) $[\text{T}]$ $[\text{F}]$

A put option struck at $100$ trades at $10$, while a put option struck at $110$ trades at $11$. Both puts have the same time to maturity. This economy admits an arbitrage.

[Hint: Consider 11 units of the first put and 10 units of the second put]

\[ 10 \times 110 > 11 \times 110 \quad \text{to avoid arbitrage} \]

\[ 10 \times 11 = 110 \quad \text{and} \quad 11 \times 10 = 110 \]

\[ \therefore \quad \text{it is an arbitrage.} \]

2. Sketch the option price as a function of the current spot-level for maturities of $T = 0$, $T = 1$ month and $T = 1$ year for

(a) $[\text{S}]$ call option

[draw the three curves on the same graph, clearly label them and any interesting points.]

(b) $[\text{S}]$ bear spread option. This option can be viewed as a long put struck at $K_2$ and a short put struck at $K_1$ ($0 < K_1 < K_2 < \infty$)

[draw the three curves on the same graph, clearly label them and any interesting points.]
3. [10] Consider an economy with the three traded assets below. Construct an arbitrage strategy.

\[ 10 \alpha + 6 \beta = 110 \]
\[ 0 \alpha + 4 \beta = 50 \Rightarrow 9 \alpha + 6 \beta = 75 \]

\[ \alpha = 35 \quad \beta = -40 \]

and long 35 B + short 40 C

has value 150, i.e. asset A is too cheap

long A, short 35, long 40 C
4. Consider an economy with a defaultable stock and bond (interest rates are 0 in all states of the world):

![Defaultable Stock price tree and Defaultable Bond price tree]

(a) Determine the risk-neutral default probabilities shown in the diagram \( a_u, a_d, b_u, b_d, c_u, c_d \).

[Hint: It is easiest if you find \( b_d \) and \( c_d \) by using the default bond tree, then find \( b_u \) and \( c_u \) from the default stock tree, and finally find \( a_d \) and \( a_u \) from both trees.]

From default bond:

\[
9.5 = (1-\lambda_u) 100 \quad \Rightarrow \quad \lambda_u = \frac{1}{120}
\]

\[
9.5 = (1-\lambda_d) 100 \quad \Rightarrow \quad \lambda_d = \frac{1}{120}
\]

\[
90.25 = (1-\lambda) 100 \quad \Rightarrow \quad \lambda = \frac{1}{120}
\]

From default stock:

\[
(1-\lambda_u) (\frac{q_d}{120} + (1-q_u)100) = 110
\]

\[
\Rightarrow 100 + 20 \frac{q_u}{120} = 110 \cdot \frac{20}{120}
\]
\( q_u = \frac{110}{19} - 5 = \frac{15}{19} \)

\[(1-\lambda)\left(q_u 100 + (1-q_u) 80\right) = 90 \]

\( \Rightarrow 80 + 20 q_u = 90 \cdot \frac{20}{19} \)

\( \Rightarrow q_u = \frac{90}{19} - 4 = \frac{14}{19} \)

\[(1-\lambda)\left(q_d 110 + (1-q_d) 90\right) = 100 \]

\( \Rightarrow 90 + 20 q_d = 100 \cdot \frac{20}{19} \)

\( \Rightarrow q_d = \frac{100}{19} - \frac{9}{2} = \frac{20}{38} \)

(b) Value an American put option with the stock as underlier, strike of 100 and maturity of two periods.

(c) Sebastian Jaimungal, 2009

\[ P_u^* = \frac{1}{20} \times 100 = 5 > K_u = 0 \quad \therefore P_u = 5 \]

\[ P_d^* = \left(\frac{1}{20} \times 100 + \frac{19}{20} \times \frac{4}{19} \times 20\right) \times 95 = 8.55 < K_u = 10 \quad \therefore P_d = 10 \]
\[ P = \left( \frac{1}{2} \times 100 + \frac{1}{2} \left( \frac{20}{38} \times 5 + \frac{9}{38} \times 10 \right) \right) \frac{90.25}{95} \]
\[ = 10.33 \]

5. Consider the interest rate tree shown in the diagram below. The rates correspond to effective discounting – e.g. discounting over the first period is \(1/(1 + 0.04)\). The probabilities shown are risk-neutral probabilities.

(a) Consider a 2-year coupon-bearing bond with coupons of $5 paid every year and notional of $100. Determine the rate \(R\) such that the bond is valued at par (i.e. has current value of 100).

\[ B_u = \frac{105}{1+R} + 5, \quad B_d = \frac{105}{1.02} + 5 \]

\[ B_0 = \frac{1}{1.04} \left[ \frac{1}{2} \left( \frac{105}{1+R} + 5 \right) + \frac{1}{2} \left( \frac{105}{1.02} + 5 \right) \right] \]

\[ = 54.299 + \frac{50.48}{1+R} \]

\[ = 100 \]

\[ \Rightarrow R = 10.46\% \]

(b) Determine the price and replication strategy of a call option maturing at \(t = 1\) written on the coupon-bearing bond with strike equal to today’s price of the bond. Note: the option holder will not receive the coupon due at \(t = 1\).
6. (a) /5/ Assuming the Black-Scholes model, derive an expression for a contingent claim which pays the geometric average of the asset price at two points in time. That is, the claim pays $(S_{T_1} S_{T_2})^{\frac{1}{2}}$ at maturity $T_2$ where $0 < T_1 < T_2$.

\[
V_0 = e^{-r T_2} \mathbb{E}_\mathbb{Q}^\alpha \left[ (S_{T_1} S_{T_2})^{\frac{1}{2}} \right]
\]

\[
S_{T_1} = S_0 \ e^{(r - \frac{1}{2} \sigma^2) T_1 + \sigma \sqrt{T_1} \ Z_1}
\]

\[
S_{T_2} = S_{T_1} \ e^{(r - \frac{1}{2} \sigma^2) (T_2 - T_1) + \sigma \sqrt{T_2 - T_1} \ Z_2}
\]

\[
= S_0 \ e^{(r - \frac{1}{2} \sigma^2) T_2 + \sigma \sqrt{T_1} \ Z_1 + \sigma \sqrt{T_2 - T_1} \ Z_2}
\]

\[
Z_1, Z_2 \quad \text{iid} \quad \sim \mathcal{N}(0, 1)
\]

\[
\Rightarrow V_0 = S_0 \ e^{-r T_2 + \frac{1}{2} (r - \frac{1}{2} \sigma^2) (T_1 + T_2)} \ \mathbb{E}_\mathbb{Q}^\alpha \left[ e^{\sigma \sqrt{T_1} \ Z_1} \ e^{\frac{1}{2} \sigma \sqrt{T_2 - T_1} \ Z_2} \right]
\]

\[
= S_0 \ e^{\frac{1}{2} \sigma^2 T_1} \ e^{\frac{1}{2} \sigma^2 (T_2 - T_1)}
\]

\[
\times \ e^{\frac{1}{2} \sigma^2 \left( T_1 + T_2 \right)} \ e^{\frac{1}{8} \left( T_1 + T_2 \right) \sigma^2}
\]

\[
= S_0 \ e^{\frac{1}{2} \sigma^2 \left( T_1 + T_2 \right) + \frac{1}{8} \left( T_1 + T_2 \right) \sigma^2}
\]

\[
(\text{c) Sebastian Jaimungal, 2009})
\]
(b) [5] Assuming the Black-Scholes model, derive an expression for a “forward start digital call option”. A forward start digital call is an option which pays 1 at the maturity date $T$ if the stock price at time $T$ is larger than the stock price at time $U$. ($U < T$) Write your answer in terms of $\Phi(x):=Q(Z < x)$ where $Z$ is a standard normal random variable under the measure $Q$.

\[
V_0 = e^{-rT} \mathbb{E}_Q^\mathbb{Q} \left[ 1 \mathbb{I}_{S_T > S_U} \right]
\]

\[
\neq \mathbb{E}_Q^\mathbb{Q} \left[ 1 e^{(r-\frac{1}{2}\sigma^2)(T-U) + \sigma \sqrt{T-U} Z} \right]
\]

\[
= e^{-rT} \mathbb{Q} \left( Z > \frac{(r-\frac{1}{2}\sigma^2) \sqrt{T-U}}{\sigma} \right)
\]

\[
= e^{-rT} \Phi \left( \frac{r-\frac{1}{2}\sigma^2}{\sigma} \sqrt{T-U} \right)
\]

7. Suppose you model stock prices in a CRR like fashion. However, you assume that

\[
S_n = S_{n-1} \exp \{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t} x_n\}
\]

where $x_1, x_2, \ldots$ are iid r.v. with $\mathbb{P}(x_1 = +1) = p$ and $\mathbb{P}(x_1 = -1) = 1-p$.

(a) [5] Prove that if we force

\[
\mathbb{P}^\mathbb{P}[S_T] = S_0 e^{\mu T},
\]

\[
\mathbb{V}^\mathbb{P}[\ln(S_T/S_0)] = \sigma^2 T
\]

in the limit as $\Delta t \downarrow 0$ while $T = n\Delta t$ is held fixed. Then,

\[
p = \frac{1}{2} \left( 1 + \frac{\mu - r}{\sigma} \sqrt{\Delta t} \right) + O(\Delta t).
\]

\[
S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)\Delta t n + \sigma \sqrt{\Delta t}(x_1 + \ldots + x_n)}
\]

\[
= S_0 e^{(r-\frac{1}{2}\sigma^2)T + X}
\]
where \[ X = \sigma \sqrt{\Delta t} \left( z_1 + \cdots + z_m \right) \xrightarrow{\text{as } n \to \infty} \mathcal{N}(m, \nu^2) \]

\[
m = \sigma n \sqrt{\Delta t} (2p-1)
\]

\[
\nu^2 = \sigma^2 \Delta t n \left( 1 - (2p-1)^2 \right)
\]

\[
= \sigma^2 \Delta t \left( 1 - (2p-1)^2 \right)
\]

\[
\therefore \quad \mathbb{E}^{\sigma} [S_T] = S_0 \left( e^{r-\frac{1}{2} \sigma^2} T + \frac{1}{2} \nu^2 + m \right)
\]

\[
\therefore \quad (r-\frac{1}{2} \sigma^2) T + \frac{1}{2} \nu^2 + m = \mu T
\]

\[
\Rightarrow \quad m + \frac{\nu^2}{2} = \left( (\mu - r) + \frac{1}{2} \sigma^2 \right) T
\]

\[
\sqrt{\mathbb{E}^{\sigma} \left[ \left( \frac{S_T}{S_0} \right)^2 \right]} = \nu^2 \geq \sigma^2 T
\]

\[
\therefore \quad m = \left( (\mu - r) + \frac{1}{2} \sigma^2 \right) T - \frac{1}{2} \sigma^2 T
\]

\[
= (\mu - r) T
\]

\[
\therefore \quad \sigma n \sqrt{\Delta t} (2p-1) = (\mu - r) T
\]

\[
\Rightarrow \quad p = \frac{1}{2} \left( 1 + \frac{\mu - r}{\sigma} \sqrt{\Delta t} \right)
\]
(b) Prove that, in the limit as $\Delta t \downarrow 0$ while $T = n\Delta t$ is held fixed, the risk neutral probability in this model is (with constant rate of interest $r$)

$$q = \frac{1}{2} + O(\Delta t).$$

and that

$$\mathbb{E}^Q[S_T] = S_0 e^{rT},$$

$$\forall^Q \ln(S_T/S_0) = \sigma^2 T.$$

$$\Rightarrow S = e^{r\Delta t} \left[ q S \left(e^{\left(-\frac{1}{2}\sigma^2\right)\Delta t + \Delta \sqrt{\Delta t}} + (1-q) S \left(e^{\left(-\frac{1}{2}\sigma^2\right)\Delta t - \Delta \sqrt{\Delta t}} \right) \right) \right]$$

$$\Rightarrow q = \frac{e^{r\Delta t} - e^{\left(-\frac{1}{2}\sigma^2\right)\Delta t - \Delta \sqrt{\Delta t}}}{e^{\left(-\frac{1}{2}\sigma^2\right)\Delta t + \Delta \sqrt{\Delta t}} - e^{\left(-\frac{1}{2}\sigma^2\right)\Delta t - \Delta \sqrt{\Delta t}}}$$

$$\approx \frac{(1 + \frac{\sigma \Delta t}{2} + \cdots) - (1 + \frac{\sigma \Delta t}{2} + \frac{\sigma^2 \Delta t}{2} + \cdots)}{(1 + \frac{r-\frac{1}{2}\sigma^2}{2} \Delta t + \frac{\sigma \Delta t}{2} + \frac{\sigma^2 \Delta t}{2} + \cdots)}$$

$$= \frac{\sigma \sqrt{\Delta t} + \cdots}{2 \sigma \sqrt{\Delta t} + \cdots} = \frac{1}{2} + \cdots$$