UNIVERSITY OF TORONTO

Faculty of Arts and Science

Term Test, October 7th, 2008

ACT 460 / STA 2502

DURATION - 150 minutes

EXAMINER: Prof. S. Jaimungal

COURSE CODE (circle one): ACT 460 STA 2502

LAST NAME: ____________________________________________

FIRST NAME: ____________________________________________

STUDENT #: ____________________________________________

Each question is worth 10 points
– NOT ALL QUESTIONS ARE OF THE SAME DIFFICULTY.

Please write clearly!

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1. [10] Please indicate true or false. **no explanations required**
   -1 for incorrect answer, +2 for correct answer, 0 for blank answer.

(a) [T] [F]
    If an economy admits a strategy which costs nothing and at a future time has strictly positive values with probability greater than zero, then this economy admits an arbitrage.

(b) [T] [F]
    The price of a put option always increases with volatility.

(c) [T] [F]
    In a one-period economy, the risk-neutral branching probabilities are always uniquely determined.

(d) [T] [F]
    If $S_t$ is the price of a traded stock, then in the Black-Scholes economy, the risk-neutral expected rate of return of $S_t^2$ is equal to $2r + \sigma^2$.

(e) [T] [F]
    A put option struck at $100$ trades at $10$, while a put option struck at $110$ trades at $11$. Both puts have the same time to maturity. This economy admits an arbitrage.

    **[Hint: Consider 11 units of the first put and 10 units of the second put]**
2. Sketch the option price as a function of the current spot-level for maturities of $T = 0$, $T = 1$ month, and $T = 1$ year for

(a) $\text{[5]}$ call option

[draw the three curves on the same graph, clearly label them and any interesting points.]
(b) A [bear spread] option. This option can be viewed as a long put struck at $K_2$ and a short put struck at $K_1$ ($0 < K_1 < K_2 < \infty$)

[draw the three curves on the same graph, clearly label them and any interesting points.]
3. [10] Consider an economy with the three traded assets below. Construct an arbitrage strategy.
4. Consider the interest rate tree shown in the diagram below. The rates correspond to effective
discounting – e.g. discounting over the first period is \(1/(1 + 0.04)\). The probabilities shown are
risk-neutral probabilities.

\[
\begin{array}{cccccccc}
8\% & \leftarrow & R & \leftarrow & 4\% & \leftarrow & 2\% & \leftarrow & 0\%\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
4\% & \quad \leftarrow & \frac{1}{2} & \quad \leftarrow & 4\% & \quad \leftarrow & \frac{3}{4} & \quad \leftarrow & 0\%
\end{array}
\]

(a) [6] Consider a 2-year coupon bearing bond with coupons of $5 paid every year and notional
of $100. Determine the rate \(R\) such that the bond is valued at par (i.e. has value $100 now).
(b) [4] Determine the price of a call option maturing at $t = 1$ written on the coupon bearing bond with strike equal to today’s price of the bond. Note: the option holder will not receive the coupon due at $t = 1$. 
5. Consider an economy with a defaultable stock and bond (interest rates are 0 in all states of the world):

\[
\begin{align*}
120 \quad &\rightarrow\quad 100 \\
110 \quad &\rightarrow\quad 95 \\
90 \quad &\rightarrow\quad 80 \\
0 \quad &\rightarrow\quad 0
\end{align*}
\]

\[
\begin{align*}
90.25 \quad &\rightarrow\quad 100 \\
95 \quad &\rightarrow\quad 100 \\
0 \quad &\rightarrow\quad 0
\end{align*}
\]

Defaultable Stock price tree

Defaultable Bond price tree

(a) [5] Show that the risk-neutral default probabilities shown in the diagram are as follows:

\[
\lambda = \lambda_d = \lambda_u = \frac{1}{20}, \quad q = \frac{29}{38}, \quad q_d = \frac{14}{19}, \quad q_u = \frac{15}{19}.
\]
(b) Value an American put option with the stock as underlier, strike of 100 and maturity of two periods.
6. (a) [5] Assuming the Black-Scholes model, derive an expression for a “forward start digital call option”. A forward start digital call is an option which pays 1 at the maturity date $T$ if the stock price at time $T$ is larger than the stock price at time $U$ ($U < T$). Write your answer in terms of $\Phi(x) := Q(Z < x)$ where $Z$ is a standard normal random variable under the measure $Q$. 

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(b) [5] Assuming the Black-Scholes model, derive an expression for a contingent claim which pays the geometric average of the asset price at two points in time. That is, the claim pays \((S_{T_1}, S_{T_2})^{\frac{1}{2}}\) at maturity \(T_2\) where \(0 < T_1 < T_2\).
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7. Suppose you model stock prices in a CRR like fashion. However, you assume that

\[ S_n = S_{n-1} \exp\{(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} \, x_n\} \]

where \( x_1, x_2, \ldots \) are iid r.v. with \( \mathbb{P}(x_1 = +1) = p \) and \( \mathbb{P}(x_1 = -1) = 1 - p \).

(a) [5] Prove that if we force

\[ \mathbb{E}^\mathbb{P}[S_T] = S_0 e^{\mu T}, \]
\[ \mathbb{V}^\mathbb{P}[\ln(S_T/S_0)] = \sigma^2 T \]

in the limit as \( \Delta t \downarrow 0 \) while \( T = n\Delta t \) is held fixed. Then,

\[ p = \frac{1}{2} \left( 1 + \frac{\mu - r}{\sigma} \sqrt{\Delta t} \right) + O(\Delta t). \]
(b) Prove that, in the limit as $\Delta t \downarrow 0$ while $T = n\Delta t$ is held fixed, the risk neutral probability in this model is (with constant rate of interest $r$)

$$q = \frac{1}{2} + O(\Delta t).$$
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