ACT 455, FEB 26/09

Note Title

DISCRETE TIME MARKOV CHAIN

$$P[X_{k+1} = j \mid X_k = i] = Q_k^{(i,j)}$$

k		$k+1$
$\vdots$		$\vdots$
i	$\longrightarrow$	j
$\vdots$		$\vdots$
X_k		X_{k+1}

HOMOGENEOUS M.C. $\rightarrow$ $Q^{(i,j)}$ DOES NOT DEPEND ON k

ONE-STEP TRANSITION PROB. MATRIX

$$\underline{Q} = \begin{array}{c} \\ i \end{array} \begin{bmatrix} & & j \\ & & Q^{(i,j)} \\ & & \end{bmatrix}$$

$S = \{0, 1, 2\}$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} Q \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} \\ \underline{Q^{(1,0)}} & \underline{Q^{(1,1)}} & \underline{Q^{(1,2)}} \\ Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} \end{bmatrix} \end{array} \times \begin{array}{c} Q \\ \\ \begin{bmatrix} Q^{(0,0)} & Q^{(0,1)} & \underline{Q^{(0,2)}} \\ Q^{(1,0)} & Q^{(1,1)} & \underline{Q^{(1,2)}} \\ Q^{(2,0)} & Q^{(2,1)} & \underline{Q^{(2,2)}} \end{bmatrix} \end{array}$$

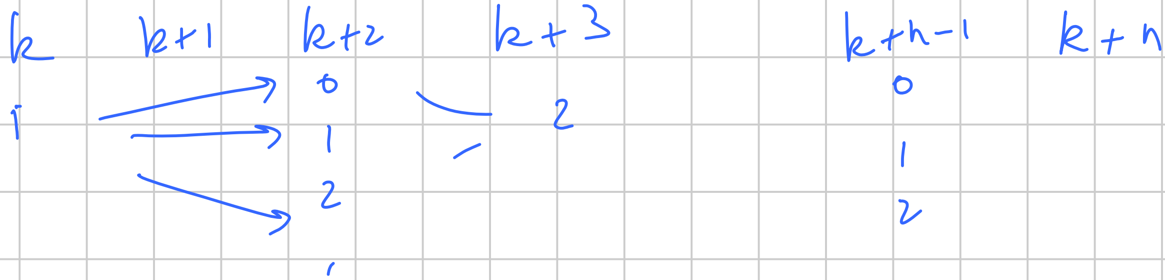
k	$k+1$	$k+2$
	0	
1	1	2
	2	

$$P[X_{k+2} = 2 \mid X_k = 1]$$

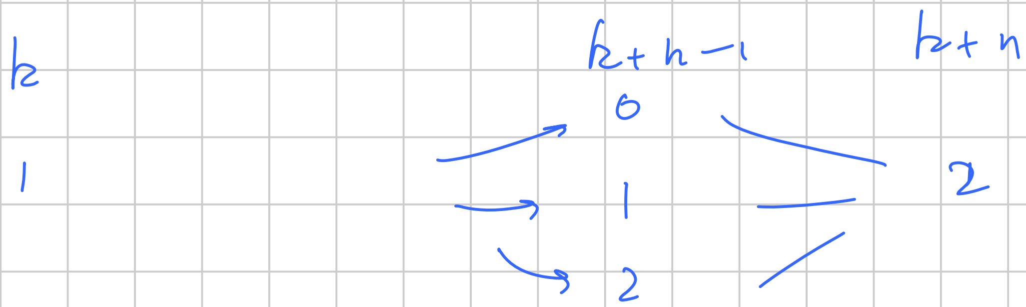
$$= Q^{(1,0)} \cdot Q^{(0,2)} + Q^{(1,1)} Q^{(1,2)} + Q^{(1,2)} Q^{(2,2)}$$

$$Q \times Q = {}_2Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} {}_2Q^{(0,0)} & {}_2Q^{(0,1)} & {}_2Q^{(0,2)} \\ \underline{{}_2Q^{(1,0)}} & \underline{{}_2Q^{(1,1)}} & \underline{{}_2Q^{(1,2)}} \\ {}_2Q^{(2,0)} & {}_2Q^{(2,1)} & {}_2Q^{(2,2)} \end{bmatrix} \end{array} \begin{bmatrix} Q^{(0,2)} \\ Q^{(1,2)} \\ Q^{(2,2)} \end{bmatrix}$$

$${}_2Q \times Q = {}_3Q \qquad {}_{n-1}Q = Q \times Q \times \cdots \times Q$$



$${}_2Q^{(1,0)}\,Q^{(0,2)} + {}_2Q^{(1,1)}\,Q^{(1,2)} + {}_2Q^{(1,2)} \cdot Q^{(2,2)}$$



n-STEP TRANSITION PROB. ARE FOUND FROM MATRIX ${}_nQ = Q \times Q \times \cdots \times Q$

BOWL CONTAINS 2 BALLS, RED OR BLUE

X = NUMBER OF BLUE IN BOWL

CHOOSE BALL AT RANDOM, REPLACE WITH BALL OF RANDOM ($\frac{1}{2}$) COLOUR

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}, \quad {}_2Q = Q \times Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{bmatrix} \end{array}$$

$${}_3Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{5}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{2} & \frac{5}{16} \end{bmatrix} \end{array}$$

$${}_kQ = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{k+1}} & \frac{1}{2} & \frac{1}{4} - \frac{1}{2^{k+1}} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} - \frac{1}{2^{k+1}} & \frac{1}{2} & \frac{1}{4} + \frac{1}{2^{k+1}} \end{bmatrix}$$

INDUCTION

OBJECTIVE:

SHOW THAT $g(k)$ IS TRUE FOR ALL INTERS $k \geq 1$

1) $g(1)$ IS TRUE

2) IF $g(n)$ TRUE THEN $g(n+1)$ IS TRUE

$${}_nQ = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{n+1}} & \cdot & \\ & & \\ & & \end{bmatrix}$$

$${}_{n+1}Q = {}_nQ \times Q = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{n+1}} & \frac{1}{2} & \frac{1}{4} - \frac{1}{2^{n+1}} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} - \frac{1}{2^{n+}} & \frac{1}{2} & \frac{1}{4} + \frac{1}{2^{n+1}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{(n+1)+1}} & & \\ & & \\ & & \end{bmatrix}$$

$${}_kQ = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{k+1}} & \frac{1}{2} & \frac{1}{4} - \frac{1}{2^{k+1}} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} - \frac{1}{2^{k+1}} & \frac{1}{2} & \frac{1}{4} + \frac{1}{2^{k+1}} \end{bmatrix} \xrightarrow{k \to \infty} \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$$

29. A machine is in one of four states (F, G, H, I) and migrates annually among them according to a Markov process with transition matrix:

	F	G	H	I
F	0.20	0.80	0.00	0.00
G	0.50	0.00	0.50	0.00
H	0.75	0.00	0.00	0.25
I	1.00	0.00	0.00	0.00

At time 0, the machine is in State F. A salvage company will pay 500 at the end of 3 years if the machine is in State F.

Assuming $v = 0.90$, calculate the actuarial present value at time 0 of this payment.

(A) 150

(B) 155

(C) 160

(D) 165

(E) 170

30. Nancy reviews the interest rates each year for a 30-year fixed mortgage issued on July 1. She models interest rate behavior by a Markov model assuming:

(i) Interest rates always change between years.

(ii) The change in any given year is dependent on the change in prior years as follows:

from year $t-3$ to year $t-2$	from year $t-2$ to year $t-1$	Probability that year t will increase from year $t-1$
Increase	Increase	0.10
Decrease	Decrease	0.20
Increase	Decrease	0.40
Decrease	Increase	0.25

She notes that interest rates decreased from year 2000 to 2001 and from year 2001 to 2002.

Calculate the probability that interest rates will decrease from year 2003 to 2004.

(A) 0.76

(B) 0.79

(C) 0.82

(D) 0.84

(E) 0.87