The asymptotic theory outlined in the nuisance parameter notes leads to the following three pivotal quantities, in the case that $\theta = (\psi, \lambda)$ and $\psi \in \mathbb{R}$:

$$
\begin{align*}
    r_p(\psi) &= \text{sign}(\hat{\psi} - \psi)\sqrt{2\{\ell_p(\psi) - \ell_p(\hat{\psi})\}}^{1/2}, \\
    r_e(\psi) &= (\hat{\psi} - \psi)j_p(\hat{\psi})^{1/2}, \\
    r_u(\psi) &= \ell_p(\psi)j_p(\hat{\psi})^{-1/2},
\end{align*}
$$

and these are all approximately standard normal pivots, under the model $f(y; \psi, \lambda)$.

[Aside: the score based pivot is not often used, because the normal approximation seems to be poor in many settings. A version of the standardized score statistic that can be useful is the version given in (8) of the nuisance parameter notes:

$$
    w_u(\psi) = U_\psi(\psi, \hat{\lambda}_\psi)^T\{i_\psi(\psi, \hat{\lambda}_\psi)\}U_\psi(\psi, \hat{\lambda}_\psi),
$$

because this requires fitting only the model with $\psi$ fixed. For example, if it were of interest to assess whether or not $\psi = 0$, i.e. whether or not the simpler model (without $\psi$) was just as good as the more complex model, then the score statistic only involves fitting the simpler model. This can be useful in some applications.]

The pivotal quantities $r_p$ and $r_e$ are illustrated in Figure 4.7 (lower) in SM (p.130), along with the profile log-likelihood function.

Here is some R code that fits a logistic regression to the Challenger shuttle data given in SM as Example 1.3. The model is $y_i \sim \text{Binomial}(m_i, p_i)$, where $m_i = 6$, and $\text{logit}(p_i) = \beta_0 + \beta_1\text{pressure}_i + \beta_2\text{temperature}_i$, $i = 1, \ldots, 23$.

```r
> library(SMPracticals)
> data(shuttle)
> head(shuttle)

   stability error sign wind magn vis use
 1       xstab    LX  pp head Light no auto
 2       xstab    LX  pp head Medium no auto
 3       xstab    LX  pp head Strong no auto
 4       xstab    LX  pp tail Light no auto
 5       xstab    LX  pp tail Medium no auto
 6       xstab    LX  pp tail Strong no auto
> ## wrong shuttle data

> data(shuttle, package = "SMPracticals")
> shuttle

   m  r temperature pressure
  1  6 0        66         50
  2  6 1        70         50
  3  6 0        69         50
  4  6 0        68         50
```
> attach(shuttle) # simplifies use of names for the next step
> shuttle.glm <- glm (cbind(r,m) ~ temperature + pressure, family = binomial)
> summary(shuttle.glm)

Call:
glm(formula = cbind(r, m) ~ temperature + pressure, family = binomial)

Deviance Residuals:
  Min 1Q Median 3Q Max
-0.9783 -0.6438 -0.5428 -0.1144 2.0898

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) 1.696161   3.405617  0.498 0.6185
temperature -0.086153   0.043549 -1.978 0.0479 *
pressure     0.007937   0.007664  1.036 0.3004
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 21.012 on 22 degrees of freedom
Residual deviance: 14.600 on 20 degrees of freedom
AIC: 34.515

Number of Fisher Scoring iterations: 5

A p-value for testing that the coefficient of temperature is zero is given (approximately) by referring the Wald statistic \( (\hat{\beta}_1 - 0) / \sqrt{\text{Var}(\hat{\beta}_1)} \) to a standard normal, and here is 0.048. Similarly the p-value for testing that \( \beta_2 = 0 \) is approximately 0.300. The likelihood ratio pivot for assessing \( \beta_1 = 0 \) is obtained by maximizing the log-likelihood function with, and without, that constraint.

> glm(cbind(r,m) ~ pressure, family=binomial)

Call: glm(formula = cbind(r, m) ~ pressure, family = binomial)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -4.371295   3.405617   -1.28  0.2044
pressure     0.009666   0.007664    1.28  0.2044

Degrees of Freedom: 22 Total (i.e. Null);  21 Residual
Null Deviance: 21.01
Residual Deviance: 18.78  AIC: 36.69
> 2*pnorm(sqrt((18.78-14.60)), lower.tail = F)  # deviance has the "2 times" built in;  
  # the outer 2 is for both tails
[1] 0.0409037

With a bit more work, it is possible to get confidence intervals based on the log-likelihood ratio pivot, and for this case the interval for \( \beta_1 \) is \((-0.1787, -0.0035)\), for the Wald pivot it is \((-0.1715, -0.0008)\).