1. Non-uniqueness of ancillary statistics. Suppose that $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ are independent and identically distributed and follow a bivariate normal distribution with $E(Y_i) = E(Z_i) = 0$, $\text{var}(Y_i) = \text{var}(Z_i) = 1$, and $\text{core}(Y_i, Z_i) = \theta$, $-1 < \theta < 1$. This is an example of a curved exponential family; it can be written in exponential family form, but the two canonical parameters are constrained to one dimension.

(a) Show that $\sum Z_i^2$ and $\sum Y_i^2$ are each ancillary for $\theta$, but that $T = \sum(Y_i^2 + Z_i^2)$ is not ancillary.

(b) Derive the first two moments of $T/\sqrt{n}$, and plot the variance of this as a function of $\theta$.

2. Logistic regression. Suppose $Y_i$ are independent Bernoulli random variables, with density

$$f(y_i) = p_i^{y_i}(1 - p_i)^{1-y_i}, \quad y = 0, 1,$$

and that

$$\log \frac{p_i}{1 - p_i} = x_i' \beta,$$

where $x_i$ and $\beta$ are vectors of length $p$.

(a) Write the joint density of $(y_1, \ldots, y_n)$ in exponential family form, and give an expression for the minimal sufficient statistic $S = (S_1, \ldots, S_p)$, say.

(b) Show that the conditional distribution of $S_j$, given $S_{(-j)}$, depends only on $\beta_j$.

3. Suppose that $Y_i$ are independent exponential random variables with $E(Y_i) = \psi \lambda_i$, and $Z_i$ are independent exponential random variables with $E(Z_i) = \psi / \lambda_i$, $i = 1, \ldots, n$.

(a) Find the maximum likelihood estimates of $\lambda_i$ and $\psi$.

(b) Show that $\hat{\psi}$ is not consistent for $\psi$ as $n \to \infty$. 
4. **Regression-scale models** Suppose $y = (y_1, \ldots, y_n)^T$ have independent components with density

$$\frac{1}{\sigma} f_0 \left( \frac{y_i - x_i^T \beta}{\sigma} \right),$$

where $f_0(\cdot)$ is a known density on $\mathbb{R}$. In HW 1 you showed that $a$ is ancillary, where $a_i = (y_i - x_i^T \tilde{\beta})/\tilde{\sigma}$, and the estimators $\tilde{\beta}$ and $\tilde{\sigma}$ are given by

$$\tilde{\beta} = (X^TX)^{-1}X^Ty, \quad \tilde{\sigma}^2 = (y - X\tilde{\beta})^T(y - X\tilde{\beta})/(n - p).$$

(In HW1 we called these $\hat{\beta}, \hat{\sigma}$, but I’ll use this notation below for the maximum likelihood estimators.)

(a) Show that under the transformation $y_i \to cy_i + x_i^Tb$, where $c > 0$, and $b = (b_1, \ldots, b_p)$ is a vector in $\mathbb{R}^p$, that we have

$$\tilde{\beta} \to c\tilde{\beta} + b, \quad \tilde{\sigma}^2 \to c^2\tilde{\sigma}^2.$$

Estimators with this property are called equivariant.

(b) Show that the associated ancillary statistic $\tilde{a} = (y - X\tilde{\beta})/\tilde{\sigma}$ is invariant under the transformation in (a).

(c) Show that the maximum likelihood estimators of $\beta$ and $\sigma$ are also equivariant, and the associated set of residuals $\hat{a} = (y - X\hat{\beta})/\hat{\sigma}$ is invariant.

(d) Deduce that the distribution of $\hat{a}$ is free of $(\beta, \sigma)$, and thus is also ancillary.

5. **Orthogonal parameters.** In a model $f(y; \theta)$ with $\theta = (\psi, \lambda)$, the component parameters $\psi$ and $\lambda$ are orthogonal (with respect to expected Fisher information) if $i_{\psi\lambda}(\theta) = 0$.

(a) Assume $y_i$ follows an exponential distribution with mean $\lambda e^{-\psi x_i}$, where $x_i$ is known. Find conditions on the sequence $\{x_i, i = 1, \ldots, n\}$ in order that $\lambda$ and $\psi$ are orthogonal with respect to expected Fisher information. Find an expression for the constrained maximum likelihood estimate $\hat{\lambda}_\psi$ and show the effect of parameter orthogonality on the form of the estimate.

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(b) Suppose that \( y_1, \ldots, y_n \) are independently normally distributed with mean
\[
E(y_i) = \frac{\alpha x_i}{\beta + x_i},
\]
where \( x_1, \ldots, x_n \) are known constants, and variance \( \sigma^2 \). This is called the Michaelis-Menten model, used in chemical kinetics. Show that \( (\alpha, \sigma^2, \chi) \) are mutually orthogonal, where
\[
\chi = \sum \frac{\alpha^3 x_i^2}{(\beta + x_i)^3}.
\]