STA3000 Sufficiency and Unbiased Estimation

This is treated in TPE at length, and also in CH §8.4. If $T = t(Y)$ is unbiased for $\theta$, and $S = s(Y)$ is sufficient for $\theta$ (in the model $f(y; \theta)$, etc.), then $W = E(T \mid S)$ is unbiased for $\theta$ and has smaller variance than $T$.

To see this,

$$
\begin{align*}
\theta &= E(T; \theta) = E\{E(T \mid S); \theta\} = E(W; \theta) \\
\text{var}(T; \theta) &= \text{var}\{E(T \mid S); \theta\} + E\{\text{var}(T \mid S); \theta\} = \text{var}(W; \theta) + E\{\text{var}(T \mid S); \theta\},
\end{align*}
$$

where by the sufficiency of $S$, neither $E(T \mid S)$ nor $\text{var}(T \mid S)$ depend on $\theta$. So $W$ is unbiased, with smaller variance unless $\text{var}(T \mid S) = 0$.

This minimum variance unbiased estimator is also unique, under the assumption that $S$ is a complete sufficient statistic. A sufficient statistic is complete if $E\{(h(S); \theta)\} = 0$ for every value $\theta$ implies $h(S) = 0$. (A sufficient statistic is boundedly complete if the same holds for bounded functions $h(\cdot)$.) This is mainly only used in establishing uniqueness of optimal tests and estimators based on sufficient statistics; it ensures that if we have two unbiased estimators of $\theta$ we end up at the same conditional estimator.

The Lehmann-Scheffé theorem says that if a sufficient statistic is boundedly complete it is minimal sufficient.

In i.i.d. sampling from an exponential family, the minimal sufficient statistic $T = t(Y)$ is complete if and only if the dimension of $T$ is the same as the dimension of $\theta$. 